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Mean-Field Critical Behavior for the Contact Process

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The contact process is a model of spread of an infectious disease. Combining with the result of ref. 1, we prove that the critical exponents take on the mean-field values for sufficiently high dimensional nearest-neighbor models and for sufficiently spread-out models with $d>4$: $\theta(\lambda) \approx \lambda - \lambda_c$ as $\lambda \downarrow \lambda_c$ and $\chi(\lambda) \approx (\lambda_c - \lambda)^{-1}$ as $\lambda \uparrow \lambda_c$, where $\theta(\lambda)$ and $\chi(\lambda)$ are the spread probability and the susceptibility of the infection respectively, and λ_c is the critical infection rate. Our results imply that the upper critical dimension for the contact process is at most 4.

KEY WORDS: Contact process; percolation; critical exponents; triangle condition; mean-field behavior; lace expansion.

1. INTRODUCTION

The contact process was introduced by Harris[8] as a model of spread of an infectious disease. This model has been studied for more than twenty-five years, and has been proved to exhibit a phase transition.

Recently it has been shown in ref. 1 that the critical behavior for the contact process becomes simple under *the triangle condition* named after a shape formed out of three infectious routes in space-time. This condition is expected to hold in spatial dimension $d>4$. By using *the lace expansion*, it has been proved that a discrete analogue of the triangle condition holds for unoriented percolation in $d \gg 6$ in ref. 7 and for oriented percolation in spatial dimension $d \gg 4$ in ref. 10. The contact process closely resembles an oriented percolation model. As in ref. 3, we can construct an oriented percolation model which converges to the contact process in a limit of the

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temporal spacing $\varepsilon \downarrow 0$. Applying the lace expansion for discrete models to the discretized contact process and taking a continuum limit, we prove that the triangle condition holds for sufficiently high dimensional *nearest-neighbor models* and for sufficiently *spread-out models* with $d > 4$, which are defined in Section 2.1. Both models are expected to exhibit the same critical behavior.

The situation for the contact process, however, is not so simple as that for discrete models. If we simply estimate the convolution equation obtained through the lace expansion as in estimating that for discrete models, we can not take a limit $\varepsilon \downarrow 0$ (see the introduction of Section 5 for more details); even if we can take a continuum limit, the expansion does not converge without any further ideas than those used to prove convergence of the lace expansion for discrete models. These potential difficulties are overcome by

1. extracting factors of ε from points where at least two infectious routes in space-time intersect,

2. selecting parameters responsible for convergence of the lace expansion, even if ε is very small.

The first point, which is inspired by the idea of ref. 1, enables us to take a continuum limit. Thanks to the second point, convergence of the lace expansion is established.

This paper is organized as follows. In Section 2.1, we introduce two types of the contact process: the nearest-neighbor model and the spread-out model. We also explain importance of analyzing critical behavior. The main results are presented in Section 2.2. They are proved by discretizing the temporal axis, and the discretization in time is performed in Section 3.1. In Section 3.2, we describe the structure of the proof of a discrete version of the main theorem. The difficult part of the proof is discussed in Section 3.3, where the lace expansion is used. The lace expansion is derived through repeated applications of the inclusion-exclusion relation, and is proved in Section 4. We establish convergence of the lace expansion and existence of a meaningful continuum limit by using the estimates of Section 5, which are obtained with the above two major points of this paper.

2. MODELS AND MAIN RESULTS

2.1. The Models

The contact process can be constructed through the graphical representation of Harris (see ref. 1 and references therein). Consider the graph

$\mathbb{Z}^d \times \mathbb{R}$, where $\mathbb{Z}^d$ and $\mathbb{R}$ denote the spatial and temporal components respectively. Along each time line $\{x\} \times \mathbb{R}$, Poisson points with intensity 1 are placed independently of the other point processes. These points stand for recovering points from the infection. For each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}$ to $\{y\} \times \mathbb{R}$, infectious arrows are drawn from x to y by a Poisson process with intensity $\lambda J_{x,y}$ independently of the other Poisson processes of arrows and points, where $\lambda \geqslant 0$ is the infection rate which is the only parameter in the contact process. We consider the following two types of the coupling constant $J_{o,x}$:

1. the nearest-neighbor model: $J_{o,x} = \mathbb{1}_{\{\|x\|_1 = 1\}}$,
2. the spread-out model: $J_{o,x} = \dfrac{\mathbb{1}_{\{0 < \|x\|_\infty \leqslant L\}}}{\sum_z \mathbb{1}_{\{0 < \|z\|_\infty \leqslant L\}}}$,

where $\|x\|_1 = \sum_j |x_j|$ and $\|x\|_\infty = \max_j |x_j|$ for $x = (x_1, x_2, ..., x_d) \in \mathbb{Z}^d$, and $\mathbb{1}_E$ denotes the indicator function of the condition E, which takes the value 1 if the condition E is satisfied, otherwise 0. The reason why we consider the spread-out model as well is presented in the end of Section 2.2.

From now on we use upper case letters for points in $\mathbb{Z}^d \times \mathbb{R}$; particularly we use O to denote the space-time origin. For $X \in \mathbb{Z}^d \times \mathbb{R}$, we write $\sigma(X)$ and $\tau(X)$ for the spatial and temporal components of X respectively. X is said to be *connected to* Y (equivalently Y is connected *from* X) if there exists a path in $\mathbb{Z}^d \times \mathbb{R}$ from X to Y using infectious arrows and temporal line segments traversed in the increasing time direction without traversing recovering points (see Fig. 1). A site X is considered to be connected to itself. We write $X \to Y$ for the connection from X to Y, and define $\mathbf{C}(X) = \{Y \in \mathbb{Z}^d \times \mathbb{R}\colon X \to Y\}$.

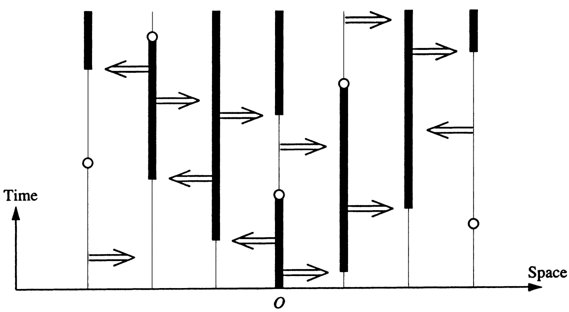

Fig. 1. The graphical representation of the nearest-neighbor model on $\mathbb{Z}^1$. The arrows in this figure are infectious arrows, and the circles are recovering points. This figure shows infectious routes from the space-time origin O.

We write $\mathbb{P}^\lambda$ and $\mathbb{E}^\lambda$ for the associated probability measure and the expectation operator. We define *the spread probability* $\theta(\lambda)$ and *the susceptibility* $\chi(\lambda)$ of the infection as

$$\theta(\lambda)=\mathbb{P}^\lambda(\mathbf{C}(O)\cap(\mathbb{Z}^d\times\{t\})\neq\varnothing,\ \forall t\geqslant 0),\quad \chi(\lambda)=\sum_{\sigma(X)}\int_0^\infty d\tau(X)\,\phi^\lambda(O,X),$$

where $\phi^\lambda(O,X)=\mathbb{P}^\lambda(O\to X)$. The generating function of the cluster size distribution is

$$M^\lambda(h)=1-\int_0^\infty dl\,e^{-hl}\,\mathbb{P}^\lambda(|\mathbf{C}(O)|=l),$$

where $|\mathbf{C}(O)|$ is the Lebesgue measure of $\mathbf{C}(O)$ in space-time.

It has been proved that there exists a positive and finite *critical infection rate* λ_c such that $\theta(\lambda)=0$ for $\lambda\leqslant\lambda_c$, $\chi(\lambda)<\infty$ for $\lambda<\lambda_c$ and

$$\begin{cases}\theta(\lambda)\geqslant C_1\,(\lambda-\lambda_c), & \text{if}\quad 0\leqslant\lambda-\lambda_c\leqslant C_1',\\ \chi(\lambda)\geqslant C_2\,(\lambda_c-\lambda)^{-1}, & \text{if}\quad 0<\lambda_c-\lambda\leqslant C_2',\\ M^{\lambda_c}(h)\geqslant C_3\,h^{1/2}, & \text{if}\quad 0<h\leqslant C_3'.\end{cases}\tag{2.1}$$

for some positive constants C_i, C_i', $i=1,2,3$ (see refs. 1–3, and references therein). The critical infection rate can also be defined as

$$\lambda_c=\sup\{\lambda\colon \Xi^\lambda(t)\to 0 \text{ exponentially as } t\uparrow\infty\},\tag{2.2}$$

where $\Xi^\lambda(t)=\sum_{X:\tau(X)=t}\phi^\lambda(O,X)$ (see ref. 11); when $\lambda>\lambda_c$, $\Xi^\lambda(t)$ does not vanish because

$$\Xi^\lambda(t)\geqslant\mathbb{P}^\lambda(\mathbf{C}(O)\cap(\mathbb{Z}^d\times\{t\})\neq\varnothing)\geqslant\theta(\lambda)>0,\qquad \forall t\geqslant 0.\tag{2.3}$$

It is expected that the observables behave in the following power law forms near and at the critical infection rate:

$$\begin{cases}\theta(\lambda)\approx(\lambda-\lambda_c)^\beta, & \text{as}\quad \lambda\downarrow\lambda_c,\\ \chi(\lambda)\approx(\lambda_c-\lambda)^{-\gamma}, & \text{as}\quad \lambda\uparrow\lambda_c,\\ M^{\lambda_c}(h)\approx h^{1/\delta}, & \text{as}\quad h\downarrow 0,\end{cases}$$

where $f(X)\approx g(X)$ as $x\to x_0$ means that there exist positive and finite constants c_1 and c_2 such that $c_1\,g(X)\leqslant f(X)\leqslant c_2\,g(X)$ for any x sufficiently close to x_0. The exponents β, γ and δ are called *the critical exponents*, which are expected to depend only on the spatial dimension d and not to depend on the type of the coupling constant. This independence is called *universality*. The inequalities in (2.1) show that the critical exponents obey the bounds $\beta \leqslant 1$, $\gamma \geqslant 1$ and $\delta \geqslant 2$, if they exist.

As in many other statistical mechanical models, it is expected (and is proved for some cases in this paper) that there exists d_c such that $\beta = 1$, $\gamma = 1$ and $\delta = 2$ as soon as d is greater than d_c. These dimension-independent values are called *the mean-field exponents*, and d_c is called *the upper critical dimension*, which is expected to be 4 for the contact process.

Barsky and Wu[1] have proved that the critical exponents take on the mean-field values under *the triangle condition* defined as

$$\sup_{\substack{X:\ \|\sigma(X)\|_1 \geqslant r,\\ \tau(X) \geqslant 0}} \nabla^{\lambda_c}(X) \to 0, \qquad \text{as} \quad r \uparrow \infty, \tag{2.4}$$

where

$$\nabla^{\lambda}(X) = \sum_{\sigma(Y),\,\sigma(Z)} \int_{\tau(X)}^{\infty} d\tau(Y) \int_{\tau(Y)}^{\infty} d\tau(Z)\, \phi^{\lambda}(O, Z)\, \phi^{\lambda}(X, Y)\, \phi^{\lambda}(Y, Z).$$

To prove the mean-field property, it thus suffices to prove that the triangle condition holds[3] above the expected upper critical dimension, 4.

2.2. The Main Results

We prove a certain condition that implies the triangle condition. We define the Fourier transform of $\phi^{\lambda}(O, X)$ as

$$\hat{\phi}^{\lambda}(K) = \sum_{\sigma(X)} \int_0^{\infty} d\tau(X)\, \phi^{\lambda}(O, X)\, e^{iK\cdot X}, \qquad K = (k, \omega) \in \Pi_d \times \mathbb{R},$$

where $\Pi_d = [-\pi, \pi]^d$ and $K \cdot X = k \cdot \sigma(X) + \omega\, \tau(X)$. Our main results can be stated as follows.

Theorem 2.1. For the nearest-neighbor model with $d \gg 4$, and for the spread-out model with $L \gg 1$ and $d > 4$, the critical exponents take on the mean-field values: $\beta = \gamma = 1$ and $\delta = 2$.

[3] For the contact process on a homogeneous d-ary tree with nearest-neighbor interaction, it has been proved in ref. 14 that the triangle condition holds when $d \geqslant 5$. This result has been extended later in ref. 12 to the case of $d \geqslant 2$.

The triangle condition (2.4) immediately follows from the following theorem and the Riemann–Lebesgue lemma, and hence we obtain Theorem 2.1, thanks to the result of ref. 1.

Theorem 2.2. Under the same condition as in Theorem 2.1, the infrared bound

$$|\hat{\phi}^{\lambda}(K)| \leqslant \frac{1}{C_\sigma \, \|k\|^2 + C_\tau \, |\omega|}, \qquad \text{uniformly in } \lambda < \lambda_c,$$

holds for some positive constants C_σ and C_τ, where $K = (k, \omega)$ and $\|k\|^2 \equiv \|k\|_2^2 = \sum_j k_j^2$.

The spread-out model is a model on a low-dimensional lattice with large coordination number, and is expected to be in the same universality class as the nearest-neighbor model. The reason why we consider the spread-out model as well is that we can prove the results for $d > 4$ by taking L to be large instead of taking d to be large as for the nearest-neighbor model (see Section 3.2 for details). Therefore the upper critical dimension d_c is expected to be at most[4] 4 by the above results.

3. PROOF OF THEOREM 2.2

We prove Theorem 2.2 along the following line:

1. constructing an oriented bond percolation model on $\mathbb{Z}^d \times \varepsilon \mathbb{Z}$ which converges to the contact process in a limit $\varepsilon \downarrow 0$ (Section 3.1),

2. applying the lace expansion to *the connectivity function* for this oriented percolation (Section 3.3),

3. estimating *the irreducible two-point functions* appearing in the lace expansion to obtain a ε-uniform infrared bound (Section 5).

3.1. Discretization

The discretized model is the following oriented bond percolation. Let $\varepsilon \in (0, 1)$ be fixed. As in the continuous-time case, we use upper case letters for points in $\mathbb{Z}^d \times \varepsilon \mathbb{Z}$. We call an ordered pair (X, Y) of sites with $\tau(X) + \varepsilon = \tau(Y)$ a *bond* from X to Y; (X, Y) is called a *temporal* bond if $\sigma(Y) = \sigma(X)$, otherwise a *spatial* bond. We call X and Y *the bottom* and

[4] A necessary condition for $d_c = 4$ can be proved by an argument analogous to that of ref. 13.

the top of (X, Y) respectively. A bond (X, Y) is *open* with probability $p(X, Y)$ defined as

$$p(X, Y) = \lambda\, \varepsilon\, \mathscr{J}_{X,Y}\, \mathbb{1}_{\{(X,Y) \text{ is a spatial bond}\}} + (1-\varepsilon)\, \mathbb{1}_{\{(X,Y) \text{ is a temporal bond}\}}, \tag{3.1}$$

where $\mathscr{J}_{X,Y} = J_{\sigma(X),\sigma(Y)}$, and *closed* with probability $1-p(X, Y)$, independently of the other bonds. Connections for these oriented percolation are made along open bonds (see Fig. 2).

We write $\mathbb{P}_\varepsilon^\lambda$ and $\mathbb{E}_\varepsilon^\lambda$ for the associated probability measure and the expectation operator. It can be proved as in ref. 3 that $\mathbb{P}_\varepsilon^\lambda$ converges weakly to $\mathbb{P}^\lambda$ as $\varepsilon \downarrow 0$. Therefore the contact process is well-approximated by the above discretized model.

As in the continuous-time case, we write $X \to Y$ for the connection from X to Y, and $\mathbf{C}(X)$ for the set of sites connected from X. We define the susceptibility as

$$\chi_\varepsilon(\lambda) = \varepsilon \sum_X \varphi_\varepsilon^\lambda(O, X),$$

where $\varphi_\varepsilon^\lambda(O, X) \equiv \mathbb{P}_\varepsilon^\lambda(O \to X)$ is the connectivity function from O to X. This discretized model also undergoes a phase transition, and we define its critical point λ_c^ε as (2.2) by replacing $\Xi^\lambda(t)$, $t \in \mathbb{R}$ with $\Xi_\varepsilon^\lambda(t) \equiv \sum_{X:\tau(X)=t} \varphi_\varepsilon^\lambda(O, X)$, $t \in \varepsilon\, \mathbb{Z}$. The Fourier transform of $\varphi_\varepsilon^\lambda(O, X)$

$$\hat{\varphi}_\varepsilon^\lambda(K) = \varepsilon \sum_X \varphi_\varepsilon^\lambda(O, X)\, e^{iK \cdot X}, \qquad K \in \Pi_d \times \frac{\Pi_1}{\varepsilon}, \tag{3.2}$$

is well-defined for $\lambda < \lambda_c^\varepsilon$ and satisfies $\chi_\varepsilon(\lambda) = \hat{\varphi}_\varepsilon^\lambda(O)$.

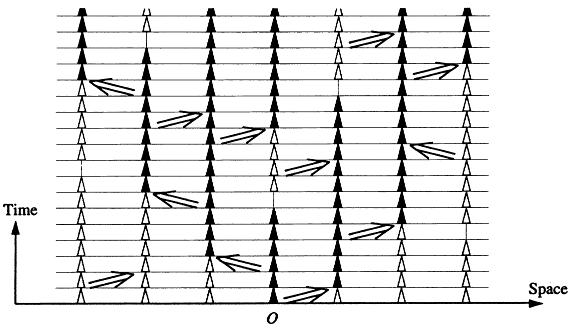

Fig. 2. This figure is obtained from Fig. 1 by discretizing the temporal axis. The arrows in this figure are open spatial bonds, and the triangles are open temporal bonds. We shade the triangles connected from O.

We can see $\lambda_c^\varepsilon \to \lambda_c$ as $\varepsilon \downarrow 0$ along the following line: for $X \in \mathbb{Z}^d \times \mathbb{R}$, we define $\phi_\varepsilon^\lambda(O, X) = \varphi_\varepsilon^\lambda(O, [X]_\varepsilon)$, where $[X]_\varepsilon = (\sigma(X), [\tau(X)/\varepsilon]\,\varepsilon) \in \mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$, and redefine $\Xi_\varepsilon^\lambda(t)$ as a function of $t \in \mathbb{R}$ by using $\phi_\varepsilon^\lambda(O, X)$ instead of $\varphi_\varepsilon^\lambda(O, X)$. For any $\lambda, t \geqslant 0$, $\Xi_\varepsilon^\lambda(t)$ converges to $\Xi^\lambda(t)$ as $\varepsilon \downarrow 0$ because $\phi_\varepsilon^\lambda(O, X) \to \phi^\lambda(O, X)$ as $\varepsilon \downarrow 0$ for any $\lambda \geqslant 0$, $X \in \mathbb{Z}^d \times \mathbb{R}$ (see ref. 14). Suppose $\liminf_{\varepsilon \downarrow 0} \lambda_c^\varepsilon < \lambda_c$. Then there exists $\lambda \in (\liminf \lambda_c^\varepsilon, \lambda_c)$ such that, in a limit $t \uparrow \infty$, $\Xi_\varepsilon^\lambda(t)$ does not converge to zero for sufficiently small ε because of the discrete version of the inequality (2.3), in spite of the exponential decay of $\Xi^\lambda(t)$, which is a contradiction. We can also derive a contradiction in another case, $\limsup_{\varepsilon \downarrow 0} \lambda_c^\varepsilon > \lambda_c$, and hence obtain $\lambda_c = \lim_{\varepsilon \downarrow 0} \lambda_c^\varepsilon$.

We obtain Theorem 2.2 by using the ε-uniform infrared bound of the following theorem and the dominated convergence theorem.

Theorem 3.1. 1. For $\varepsilon_0 < 1$, there exists $d_0 \in (4, \infty)$ such that, for the nearest-neighbor model with $d > d_0$ and $\varepsilon < \varepsilon_0$, the following infrared bound holds:

$$|\hat{\varphi}_\varepsilon^\lambda(K)| \leqslant \frac{1}{C_\sigma\,\|k\|^2 + C_\tau\,|\omega|}, \qquad \text{uniformly in } \lambda < \lambda_c^\varepsilon,$$

for some ε-independent positive constants C_σ and C_τ.

2. For $d > 4$, there exists $L_0 < \infty$ such that, for the spread-out model with $L > L_0$ and $\varepsilon < 1$, the above infrared bound holds.

3.2. Structure of the Proof of Theorem 3.1

We omit the subscript ε and the superscript λ of various quantities in the rest of this paper. We also omit the superscript ε of λ_c^ε.

The proof of Theorem 3.1 is achieved by showing that the following three statements hold.

1. For every $X \in \mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$,

$$\mathscr{K}(X) = \varepsilon^2 \sum_{\substack{Y, Z,\\ O', O''}} \frac{\mathscr{I}_{O,O'}}{|\mathscr{I}|} \frac{\mathscr{I}_{O,O''}}{|\mathscr{I}|}\, \varphi(O', Z)\, \varphi(X + O'', Y)\, \varphi(Y, Z),$$

$$\mathscr{T}(X) = \varepsilon^2 \sum_{Y, Z} \{\varphi(O, Z)\, \varphi(X, Y)\, \varphi(Y, Z) - \Delta_{O,Z}\, \Delta_{X,Y}\, \Delta_{Y,Z}\},$$

$$\mathscr{W}(X) = \varepsilon \sum_{Z} \|\sigma(Z)\|^2\, \varphi(O, Z)\, \varphi(X, Z).$$

are finite for $\lambda < \lambda_c$ and continuous in $\lambda \leqslant \lambda_c$, where $\Delta_{O,X} \equiv \sum_{n=0}^{\infty} (1-\varepsilon)^n \times \mathbb{1}_{\{X=(o,n\varepsilon)\}}$ is the probability that there is an open path from O to X made of only temporal bonds[5].

2. For $d > 4$ and $\lambda \leqslant |\mathscr{J}|^{-1} \equiv (\sum_X \mathscr{J}_{O,X})^{-1}$, there exist ε-independent finite constants $C_{\mathscr{K}}$, $C_{\mathscr{T}}$, $C_{\mathscr{W}}$ and $C'_{\mathscr{W}}$ such that

$$\mathscr{K}(X) \leqslant C_{\mathscr{K}}\, \kappa, \qquad \mathscr{T}(X) \leqslant C_{\mathscr{T}}\, \mu, \qquad \mathscr{W}(X) \leqslant \begin{cases} C_{\mathscr{W}}\, \rho, & \text{if} \quad X = O, \\ C'_{\mathscr{W}}\, \zeta, & \text{if} \quad X \neq O. \end{cases} \tag{3.3}$$

For the nearest-neighbor model, $\kappa = \mu = \rho = \zeta = d^{-1}$. For the spread-out model,

$$\kappa = L^{-d} (\ln L)^{d/2}, \qquad \mu = \kappa^{1/2}, \qquad \rho = L^{4-d} (\ln L)^d, \qquad \zeta = L^2\, \mu.$$

3. We define $\mathfrak{P}_a$ for the following set of inequalities:

$$\lambda\, |\mathscr{J}| \leqslant a, \qquad \mathscr{K}(X) \leqslant a\, C_{\mathscr{K}}\, \kappa, \qquad \mathscr{T}(X) \leqslant a\, C_{\mathscr{T}}\, \mu,$$
$$\mathscr{W}(X) \leqslant \begin{cases} a\, C_{\mathscr{W}}\, \rho, & \text{if} \quad X = O, \\ a\, C'_{\mathscr{W}}\, \zeta, & \text{if} \quad X \neq O. \end{cases} \tag{3.4}$$

Fix $\lambda \in [|\mathscr{J}|^{-1}, \lambda_c)$ arbitrarily. If we take d or L to be sufficiently large, then $\mathfrak{P}_4$ implies $\mathfrak{P}_3$.

The first and second statements imply that $\mathscr{K}(X)$, $\mathscr{T}(X)$ and $\mathscr{W}(X)$ are small satisfying $\mathfrak{P}_3$ for small λ and continuous up to λ_c for every $X \in \mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$. The third statement implies that, for sufficiently large d or L, there is a forbidden region in the graph of $\mathscr{K}(X)$, $\mathscr{T}(X)$ and $\mathscr{W}(X)$. Therefore we can see that the stronger inequalities in $\mathfrak{P}_3$ indeed hold at λ_c. The infrared bound is obtained in the course of the proof of the third statement.

The finiteness of $\mathscr{K}(X)$, $\mathscr{T}(X)$ and $\mathscr{W}(X)$ for $\lambda < \lambda_c$ in the first statement follows from the discrete version of the identity (2.2), and their continuity in $\lambda \leqslant \lambda_c$ follows from the continuity of $\varphi(O, X)$ in $\lambda \geqslant 0$ and the monotone convergence theorem.

[5] In refs. 7 and 10, $\delta_{O,X} \equiv \mathbb{1}_{\{O=X\}}$ has been used instead of $\Delta_{O,X}$ in defining $\mathscr{T}(X)$. We must use $\Delta_{O,X}$ for the models considered in this paper to obtain a desirable bound on $\mathscr{T}(X)$ in large d or L (see Section 3.4).

The Green function $G(O, X)$ for the random walk on $\mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$ is defined by the transition probability

$$p_{\mathrm{G}}(O, X) = \frac{\varepsilon\, \mathscr{J}_{O,X}}{|\mathscr{J}|}\, \mathbb{1}_{\{(O,X)\text{ is a spatial bond}\}} + (1-\varepsilon)\, \mathbb{1}_{\{(O,X)\text{ is a temporal bond}\}}, \tag{3.5}$$

and satisfies the convolution equation

$$G(O, X) = \delta_{O,X} + \sum_{Z} p_{\mathrm{G}}(O, Z)\, G(Z, X). \tag{3.6}$$

We can see $\varphi(O, X) \leqslant G(O, X)$ in $\lambda \leqslant |\mathscr{J}|^{-1}$ by comparing (3.1) and (3.5) and following the way of proving Lemma 4.1 of ref. 7. The second statement follows from this inequality and bounds on the quantities $\mathscr{K}_{\mathrm{G}}(X)$, $\mathscr{T}_{\mathrm{G}}(X)$ and $\mathscr{W}_{\mathrm{G}}(X)$, which are proved in Section 3.4, defined by replacing φ with G in the definition of $\mathscr{K}(X)$, $\mathscr{T}(X)$ and $\mathscr{W}(X)$ respectively.

The difficult part of the proof of Theorem 3.1 is to obtain the third statement, and here the lace expansion is used.

3.3. Bootstrapping Argument

We describe the outline for deriving the stronger inequalities in $\mathfrak{P}_3$ under the weaker inequalities in $\mathfrak{P}_4$. We suppose $\varepsilon \leqslant 1/3$ in the rest of this paper; the remaining case of $\varepsilon > 1/3$ can be proved with slight modifications, and we omit this case in this paper.

We obtain in Section 4 the convolution equation for the connectivity function

$$\begin{aligned}\varphi(O, X) = \Gamma_N(O, X) + \sum_{(U,V)} \Gamma_N(O, U)\, p(U, V)\, \varphi(V, X)\\ + (-1)^{N+1} R_{N+1}(O, X),\end{aligned}$$

where $\Gamma_N(O, X) = \sum_{n=0}^{N} (-1)^n g_n(O, X)$, $g_n(O, X)$ is the irreducible two-point function, and $R_N(O, X)$ is a remainder, which are explained in Section 4. Taking the Fourier transform of the above equation, we obtain

$$\hat{\varphi}(K) = \frac{\hat{\Gamma}_N(K) + (-1)^{N+1} \hat{R}_{N+1}(K)}{1 - P(K)\, \hat{\Gamma}_N(K)/\varepsilon}, \tag{3.7}$$

where

$$P(K) = \sum_{X} p(O, X)\, e^{iK\cdot X} = \{1 - \varepsilon + \lambda\, |\mathscr{J}|\, \varepsilon\, D(k)\}\, e^{i\omega\varepsilon},$$

with $D(k)=\sum_x J_{o,x}\, e^{ik\cdot x}/|\mathscr{J}|$, which is $d^{-1}\sum_{j=1}^d \cos k_j$ for the nearest-neighbor model. We put hats as $\hat{\Gamma}_N(K)$ and $\hat{R}_{N+1}(K)$ to describe the Fourier transforms of $\Gamma_N(O,X)$ and $R_{N+1}(O,X)$ respectively, as we defined $\hat{\varphi}(K)$ in (3.2). Under the inequalities in $\mathfrak{P}_4$, we obtain in Section 5.1 the following bounds on the irreducible two-point functions:

$$|\hat{g}_n(K)| \leqslant \begin{cases} \varepsilon + \mathcal{O}(\kappa)\,\varepsilon^2, & \text{for} \quad n=0, \\ \mathcal{O}(\kappa)\,\mathcal{O}(\mu)^{n-1}\,\varepsilon^2, & \text{for} \quad n\geqslant 1. \end{cases} \tag{3.8}$$

In Section 5.2, we obtain the inequality

$$|\hat{R}_{N+1}(K)| \leqslant \hat{g}_N(O)\,|p|\,\chi(\lambda)/\varepsilon, \tag{3.9}$$

where $|p|=P(O)$. Therefore for sufficiently large d or L, we can take N to infinity to obtain $\hat{\Gamma}_N(K)\to\hat{\Gamma}(K)=\varepsilon+\mathcal{O}(\kappa)\,\varepsilon^2$ and $\hat{R}_{N+1}(K)\to 0$. These facts enable us to rewrite $\hat{\varphi}(K)$ as

$$\begin{aligned}\hat{\varphi}(K) &= \frac{\hat{\Gamma}(K)}{1-\lambda\,|\mathscr{J}|\,D(k)\,e^{i\omega\varepsilon}\,\hat{\Gamma}(K)-(1-\varepsilon)\,e^{i\omega\varepsilon}\,\hat{\Gamma}(K)/\varepsilon} \\ &= \frac{\hat{\Gamma}(K)/\hat{\Gamma}(O)}{\lambda\,|\mathscr{J}|\,\{1-D(k)\}+P(k,0)\,\dfrac{1-e^{i\omega\varepsilon}}{\varepsilon}+P(K)\,\dfrac{\hat{\Gamma}(O)-\hat{\Gamma}(K)}{\varepsilon\hat{\Gamma}(O)}+\chi(\lambda)^{-1}},\end{aligned} \tag{3.10}$$

where we used the identity

$$\chi(\lambda)=\hat{\varphi}(O)=\frac{\hat{\Gamma}(O)}{1-\lambda\,|\mathscr{J}|\,\hat{\Gamma}(O)-(1-\varepsilon)\,\hat{\Gamma}(O)/\varepsilon}.$$

The numerator is non-negative for sufficiently large d or L, and thus the denominator is also non-negative. Therefore[6]

$$\lambda\,|\mathscr{J}| \leqslant \frac{1-(1-\varepsilon)\,\hat{\Gamma}(O)/\varepsilon}{\hat{\Gamma}(O)} = 1+\frac{\varepsilon-\hat{\Gamma}(O)}{\varepsilon\,\hat{\Gamma}(O)} \leqslant 1+\mathcal{O}(\kappa). \tag{3.11}$$

[6] In fact, the inequality (3.11) holds up to λ_c, if d is sufficiently large or if $d>4$ and L is sufficiently large. Since we know for the spread-out model that $\lambda_c=\lambda_c\,|\mathscr{J}|\geqslant 1$, we obtain the inequality $0\leqslant\lambda_c-1\leqslant\mathcal{O}(\kappa)$, which is weaker than the result $\lambda_c-1\approx L^{-d}$ for $d\geqslant 3$ of ref. 4. We can find in ref. 4 the estimates for $d=1, 2$ as well.

Next we estimate the denominator of the right side of (3.10). By $\varepsilon \leqslant 1/3$ and (3.11), we obtain

$$P(k, 0) \geqslant 1-\varepsilon-\lambda\,|\mathscr{I}|\,\varepsilon \geqslant \tfrac{1}{3}\,\{1-\mathcal{O}(\kappa)\},$$

and thus obtain $P(k, 0) \geqslant 0$ for sufficiently large d or L. Together with the trivial inequalities

$$\lambda\,|\mathscr{I}| \geqslant 1, \qquad \chi(\lambda)^{-1} \geqslant 0, \qquad 1-D(k) \geqslant 0, \qquad \Re[1-e^{i\omega\varepsilon}] \geqslant 0,$$

we obtain

$$\begin{aligned}
&\left|\lambda\,|\mathscr{I}|\,\{1-D(k)\}+P(k,0)\,\frac{1-e^{i\omega\varepsilon}}{\varepsilon}+P(K)\,\frac{\hat{\Gamma}(O)-\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(O)}+\chi(\lambda)^{-1}\right| \\
&\quad\geqslant \left|\lambda\,|\mathscr{I}|\,\{1-D(k)\}+P(k,0)\,\frac{1-e^{i\omega\varepsilon}}{\varepsilon}+\chi(\lambda)^{-1}\right|-\left|P(K)\,\frac{\hat{\Gamma}(O)-\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(O)}\right| \\
&\quad\geqslant \left\{\left|\lambda\,|\mathscr{I}|\,\{1-D(k)\}+P(k,0)\,\frac{1-e^{i\omega\varepsilon}}{\varepsilon}\right|^2+\chi(\lambda)^{-2}\right\}^{1/2}-\left|P(K)\,\frac{\hat{\Gamma}(O)-\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(O)}\right| \\
&\quad\geqslant \left\{\{1-D(k)\}^2+P(k,0)^2\left|\frac{1-e^{i\omega\varepsilon}}{\varepsilon}\right|^2\right\}^{1/2}-\left|P(K)\,\frac{\hat{\Gamma}(O)-\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(O)}\right|. \qquad (3.12)
\end{aligned}$$

In Section 5.3, we obtain

$$|\hat{\Gamma}(O)-\hat{\Gamma}(K)| \leqslant \sum_n \left(\frac{\|k\|^2}{2\,d}\,\mathscr{G}_n^\sigma+|\omega|\,\mathscr{G}_n^\tau\right), \qquad (3.13)$$

where $\mathscr{G}_n^\sigma=\varepsilon\sum_X\|\sigma(X)\|^2\,g_n(O,X)$ and $\mathscr{G}_n^\tau=\varepsilon\sum_X\tau(X)\,g_n(O,X)$. We show in Section 5.3 that, under the inequalities in $\mathfrak{P}_4$,

$$\begin{aligned}
\mathscr{G}_n^\tau &\leqslant \begin{cases}\mathcal{O}(\kappa)\,\varepsilon^2, & \text{for } n=0,\\ n\,\mathcal{O}(\kappa)\,\mathcal{O}(\mu)^{n-1}\,\varepsilon^2, & \text{for } n\geqslant 1,\end{cases} \\
\mathscr{G}_n^\sigma &\leqslant \begin{cases}\mathcal{O}(\zeta)\,\varepsilon^2, & \text{for } n=0,\\ n^2\,\mathcal{O}(\zeta)\,\mathcal{O}(\mu)^{n-1}\,\varepsilon^2, & \text{for } n\geqslant 1,\end{cases}
\end{aligned} \qquad (3.14)$$

and thus $\sum_n \mathscr{G}_n^\tau \leqslant \mathcal{O}(\kappa)\,\varepsilon^2$ and $\sum_n \mathscr{G}_n^\sigma \leqslant \mathcal{O}(\zeta)\,\varepsilon^2$ for sufficiently large d or L. By using (3.13) and

$$1-D(k) \geqslant \frac{\|k\|^2}{2\,\pi^2\,d}, \qquad |1-e^{i\omega\varepsilon}| \geqslant \frac{2\,\varepsilon}{\pi}\,|\omega|,$$

the right side of (3.12) is bounded from below by

$$\frac{1}{\sqrt{2}}\left\{1-D(k)+\frac{2}{\pi}P(k,0)\,|\omega|\right\}-\left|\frac{P(K)}{\varepsilon\,\hat{\Gamma}(O)}\right|\sum_n\left(\frac{\|k\|^2}{2d}\,\mathcal{G}^{\sigma}_n+|\omega|\,\mathcal{G}^{\tau}_n\right)$$
$$\geqslant\frac{1}{\sqrt{2}}\left\{A_\sigma\,\{1-D(k)\}+\frac{2}{3\pi}A_\tau\,|\omega|\right\},\tag{3.15}$$

where

$$A_\sigma=1-\sqrt{2}\,\pi^2\left|\frac{P(K)}{\varepsilon\,\hat{\Gamma}(O)}\right|\sum_n\mathcal{G}^{\sigma}_n,\qquad A_\tau=3\,P(k,0)-\frac{3}{\sqrt{2}}\,\pi\left|\frac{P(K)}{\varepsilon\,\hat{\Gamma}(O)}\right|\sum_n\mathcal{G}^{\tau}_n.$$

For sufficiently large d or L, $A_\sigma\geqslant 1-\mathcal{O}(\zeta)$ and $A_\tau\geqslant 1-\mathcal{O}(\kappa)$. Following the way of bounding $\mathcal{T}_{\mathrm{G}}(X)$ presented in the next section, we use the inequalities (3.11) and (3.15) to obtain

$$\bar{\mathcal{K}}\equiv\sup_X\mathcal{K}(X)\leqslant C_{\mathcal{K}}\,\kappa+\mathcal{O}(\kappa\,\zeta),\qquad\bar{\mathcal{T}}\equiv\sup_X\mathcal{T}(X)\leqslant 2\,C_{\mathcal{T}}\,\mu+\mathcal{O}(\mu\,\zeta).\tag{3.16}$$

Finally we bound $\mathcal{W}(X)$. Integrating by parts, we have

$$\mathcal{W}(X)=\begin{cases}\displaystyle\sum_{j=1}^{d}\int_{\Pi_d\times\frac{\Pi_1}{\varepsilon}}\frac{dK}{(2\pi)^{d+1}}\,|\partial_j\hat{\varphi}(K)|^2, & \text{if}\quad X=O,\\[2ex] \displaystyle-\sum_{j=1}^{d}\int_{\Pi_d\times\frac{\Pi_1}{\varepsilon}}\frac{dK}{(2\pi)^{d+1}}\,\{\partial_j^2\hat{\varphi}(K)\}\,\hat{\varphi}(-K)\,e^{-iK\cdot X}, & \text{if}\quad X\neq O,\end{cases}$$

where $\partial_j=\partial/\partial k_j$. We differentiate $\hat{\varphi}(K)$ in (3.10) to obtain

$$\partial_j\hat{\varphi}(K)=\hat{\varphi}(K)\,\frac{\partial_j\hat{\Gamma}(K)}{\hat{\Gamma}(K)}+\hat{\varphi}(K)^2\left\{\lambda\,|\mathcal{J}|\,e^{i\omega\varepsilon}\,\partial_jD(k)+\frac{P(K)\,\partial_j\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(K)}\right\},$$

$$\partial_j^2\hat{\varphi}(K)=\hat{\varphi}(K)\,\frac{\partial_j^2\hat{\Gamma}(K)}{\hat{\Gamma}(K)}+2\hat{\varphi}(K)^2\,\frac{\partial_j\hat{\Gamma}(K)}{\hat{\Gamma}(K)}\left\{\lambda\,|\mathcal{J}|\,e^{i\omega\varepsilon}\,\partial_jD(k)+\frac{P(K)\,\partial_j\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(K)}\right\}$$
$$+\hat{\varphi}(K)^2\left\{\lambda\,|\mathcal{J}|\,e^{i\omega\varepsilon}\,\partial_j^2D(k)+2\,\lambda\,|\mathcal{J}|\,e^{i\omega\varepsilon}\,\partial_jD(k)\,\frac{\partial_j\hat{\Gamma}(K)}{\hat{\Gamma}(K)}\right.$$
$$\left.+\frac{P(K)\,\partial_j^2\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(K)}\right\}+2\,\hat{\varphi}(K)^3\left\{\lambda\,|\mathcal{J}|\,e^{i\omega\varepsilon}\,\partial_jD(k)+\frac{P(K)\,\partial_j\hat{\Gamma}(K)}{\varepsilon\,\hat{\Gamma}(K)}\right\}^2.$$

Thanks to the estimates of Section 5.3, $\hat{\Gamma}(K)$ is differentiable term by term. Together with the spatial symmetry of $g_n(O, X)$, $|\partial_j^s \hat{\Gamma}(K)| \leqslant d^{-1} \sum_n \mathcal{G}_n^\sigma$ for $s = 1, 2$. Following the way of bounding $\mathcal{W}_G(X)$ presented in the next section, and using the mean value theorem as in Section 4.3.3(c) of ref. 7 and the inequalities (3.11), (3.15) and (3.16), we obtain

$$\mathcal{W}(O) \leqslant 2\, C_{\mathcal{W}}\, \rho + \mathcal{O}(\rho\, \zeta) + \frac{1}{d}\, \mathcal{O}(\zeta^2), \tag{3.17}$$

and the stronger inequality for $\mathcal{W}(X)$ with $X \neq O$ by choosing $C'_{\mathcal{W}}$ sufficiently large depending only on $C_{\mathcal{K}}$, $C_{\mathcal{T}}$ and $C_{\mathcal{W}}$.

We have seen that $\mathfrak{P}_4$ implies $\mathfrak{P}_3$ if d or L is sufficiently large. The proof of the third statement in Section 3.2 is now completed, assuming estimates of Sections 4 and 5. The infrared bound has been obtained in (3.15).

3.4. Bounds on the Gaussian Quantities

We prove that the quantities $\mathcal{K}_G(X)$, $\mathcal{T}_G(X)$ and $\mathcal{W}_G(X)$ are bounded as in (3.3).

We begin with estimating $\mathcal{T}_G(X)$. Using the identity

$$\begin{aligned} &|\hat{G}(K)|^2\, \hat{G}(K) - |\hat{\Delta}(K)|^2\, \hat{\Delta}(K) \\ &\quad = \hat{G}(K)^* \{\hat{G}(K) - \hat{\Delta}(K)\}^2 + 2\, |\hat{G}(K) - \hat{\Delta}(K)|^2\, \hat{\Delta}(K) \\ &\qquad + 2\, \{\hat{G}(K) - \hat{\Delta}(K)\}\, |\hat{\Delta}(K)|^2 + \{\hat{G}(K) - \hat{\Delta}(K)\}^* \{\hat{\Delta}(K)\}^2, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{T}_G(X) \leqslant & \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\,\pi)^{d+1}} \left\{ |\hat{G}(K)|\, |\hat{G}(K) - \hat{\Delta}(K)|^2 + 2\, |\hat{G}(K) - \hat{\Delta}(K)|^2\, |\hat{\Delta}(K)| \right\} \\ & + 2\, \varepsilon^2 \sum_{Y, Z} \Delta_{O,Z}\, \{G(X, Y) - \Delta_{X,Y}\}\, \Delta_{Y,Z} \\ & + \varepsilon^2 \sum_{Y, Z} \{G(O, Z) - \Delta_{O,Z}\}\, \Delta_{X,Y}\, \Delta_{Y,Z}. \end{aligned} \tag{3.18}$$

First we estimate the integral of (3.18). Taking the Fourier transform of the convolution equation (3.6), we have

$$\hat{G}(K) = \frac{\varepsilon}{1 - P_G(K)} = \frac{1}{1 - D(k) + P_G(k, 0)\, \dfrac{1 - e^{i\omega\varepsilon}}{\varepsilon}}, \tag{3.19}$$

where $P_G(K) = \sum_X p_G(O, X)\, e^{iK \cdot X}$. By using the trivial inequalities

$$0 \leqslant 1 - D(k) \leqslant 2, \qquad \Re[1 - e^{i\omega\varepsilon}] \geqslant 0, \qquad |1 - e^{i\omega\varepsilon}| \geqslant \frac{2\,\varepsilon}{\pi}\,|\omega|,$$

$$P_G(k, 0) \geqslant 1 - 2\,\varepsilon \geqslant \frac{1}{3},$$

the integral of $|\hat{G}(K)|\,|\hat{G}(K) - \hat{\Delta}(K)|^2$ in (3.18) is bounded by

$$\int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\,\pi)^{d+1}} \frac{D(k)^2}{\left\{\frac{1}{\sqrt{2}}\left\{1 - D(k) + \frac{2}{3\pi}\,|\omega|\right\}\right\}^3} \leqslant \int_{\Pi_d} \frac{d^dk}{(2\,\pi)^d} \frac{9\,D(k)^2}{\sqrt{2}\,\{1 - D(k)\}^2}, \tag{3.20}$$

and the integral of $|\hat{G}(K) - \hat{\Delta}(K)|^2\,|\hat{\Delta}(K)|$ in (3.18) is bounded by

$$\int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\,\pi)^{d+1}} \frac{D(k)^2}{\left\{\frac{1}{\sqrt{2}}\left\{1 - D(k) + \frac{2}{3\pi}\,|\omega|\right\}\right\}^2} \leqslant \int_{\Pi_d} \frac{d^dk}{(2\,\pi)^d} \frac{3\,D(k)^2}{1 - D(k)}. \tag{3.21}$$

It has been proved in Appendix A of ref. 9 that the right sides of (3.20) and (3.21) are bounded by a d-independent multiple of $(d-4)^{-1}$ for the nearest-neighbor model, or by an L-independent multiple of $L^{-d}\,(\ln L)^{d/2}$ for the spread-out model.

Since $\mathscr{K}_G(X)$ is bounded by (3.20), this completes the proof of the bound on $\mathscr{K}_G(X)$.

Next we estimate the second sum in (3.18)

$$\varepsilon^2 \sum_{Y, Z} \{G(O, Z) - \Delta_{O,Z}\}\, \Delta_{X,Y}\, \Delta_{Y,Z}.$$

The first sum of (3.18) can be estimated in the same way. Using the Schwarz inequality, we have

$$\begin{aligned} &\varepsilon^2 \sum_{Y, Z} \{G(O, Z) - \Delta_{O,Z}\}\, \Delta_{X,Y}\, \Delta_{Y,Z} \\ &\quad \leqslant \left\{\varepsilon \sum_{m=0}^{\infty} \left(\varepsilon \sum_{n=0}^{m} \Delta_{X, X+n\Theta}\, \Delta_{X+n\Theta, X+m\Theta}\right)^2\right\}^{1/2} \\ &\quad\quad \times \left\{\varepsilon \sum_{m=0}^{\infty} \{G(O, X+m\,\Theta) - \Delta_{O, X+m\Theta}\}^2\right\}^{1/2}, \end{aligned} \tag{3.22}$$

where $\Theta=(o,\varepsilon)$. The quantity in the former square root sign is bounded by

$$\varepsilon^3\sum_{m=0}^{\infty}(m+1)^2(1-\varepsilon)^{2m}=\frac{1+(1-\varepsilon)^2}{(2-\varepsilon)^3}<1.$$

The sum in the latter square root sign of (3.22) must be estimated separately depending on whether $\sigma(X)$ is equal to o or not. If $\sigma(X)\neq o$, then we use the spatial symmetry of the two-point function to obtain

$$2d\,\varepsilon\sum_{m=0}^{\infty}\{G(O,X+m\,\Theta)-\Delta_{O,X+m\Theta}\}^2\leqslant\varepsilon\sum_{Z}\{G(O,Z)-\Delta_{O,Z}\}^2, \tag{3.23}$$

which is bounded by the right side of (3.21). If on the other hand $\sigma(X)=o$, we can not directly use the spatial symmetry of the two-point function as above. However if a random walker reaches $X+m\,\Theta=(o,\tau(X)+m\,\varepsilon)$ from O by taking at least one spatial step, then there exist $n<m+\tau(X)/\varepsilon$ and $y\in\mathbb{Z}^d$ with $J_{o,y}>0$ such that the random walker taking only temporal steps to $(o,n\,\varepsilon)$ takes the first spatial step to $(y,(n+1)\,\varepsilon)$ and reaches $(o,\tau(X)+m\,\varepsilon)$. Therefore

$$\begin{aligned}G(O,X+m\,\Theta)-\Delta_{O,X+m\Theta}\leqslant&\sum_{n=0}^{m+\tau(X)/\varepsilon}(1-\varepsilon)^n\sum_{Y}\frac{\varepsilon\,\mathscr{J}_{O,Y}}{|\mathscr{J}|}\\&\times\{G(Y,X+(m-n)\,\Theta)-\Delta_{Y,X+(m-n)\,\Theta}\}.\end{aligned}$$

Using this inequality and the Schwarz inequality, we have

$$\begin{aligned}&\varepsilon\sum_{m=0}^{\infty}\{G(O,X+m\,\Theta)-\Delta_{O,X+m\Theta}\}^2\\&\quad\leqslant\sum_{n,n'}(1-\varepsilon)^{n+n'}\sum_{Y,Y'}\frac{\varepsilon\,\mathscr{J}_{O,Y}}{|\mathscr{J}|}\frac{\varepsilon\,\mathscr{J}_{O,Y'}}{|\mathscr{J}|}\\&\qquad\times\Big\{\varepsilon\sum_{m}\{G(Y,X+(m-n)\,\Theta)-\Delta_{Y,X+(m-n)\Theta}\}\\&\qquad\times\{G(Y',X+(m-n')\,\Theta)-\Delta_{Y',X+(m-n')\Theta}\}\Big\}\\&\quad\leqslant\bigg\{\varepsilon\sum_{n}(1-\varepsilon)^n\sum_{Y}\frac{\mathscr{J}_{O,Y}}{|\mathscr{J}|}\Big\{\varepsilon\sum_{m}\{G(Y,X+(m-n)\,\Theta)-\Delta_{Y,X+(m-n)\Theta}\}^2\Big\}^{1/2}\bigg\}^2.\end{aligned}$$

Now we can use the spatial symmetry of the two-point function as in (3.23). The proof of bounding $\mathscr{T}_G(X)$ is completed.

Finally we bound $\mathscr{W}_G(X)$. Integrating by parts, we have

$$\mathscr{W}_G(X) = \begin{cases} \sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} |\partial_j \hat{G}(K)|^2, & \text{if} \quad X = O, \\ -\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \{\partial_j^2 \hat{G}(K)\} \, \hat{G}(-K) \, e^{-iK \cdot X}, & \text{if} \quad X \neq O. \end{cases}$$

We differentiate the right side of (3.19) to obtain

$$\begin{aligned} \partial_j \hat{G}(K) &= \hat{G}(K)^2 \, e^{i\omega\varepsilon} \, \partial_j D(k), \\ \partial_j^2 \hat{G}(K) &= 2 \, \hat{G}(K)^3 \, \{e^{i\omega\varepsilon} \, \partial_j D(k)\}^2 + \hat{G}(K)^2 \, e^{i\omega\varepsilon} \, \partial_j^2 D(k). \end{aligned} \tag{3.24}$$

Therefore $\mathscr{W}_G(O)$ and the contribution to $\mathscr{W}_G(X)$ with $X \neq O$ from $\hat{G}(K)^3 \, \{e^{i\omega\varepsilon} \, \partial_j D(k)\}^2$ are bounded by

$$\begin{aligned} &\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \frac{\{\partial_j D(k)\}^2}{\left\{ \frac{1}{\sqrt{2}} \left\{ 1 - D(k) + \frac{2}{3\pi} |\omega| \right\} \right\}^4} \\ &\leqslant \sum_{j=1}^{d} \int_{\Pi_d} \frac{d^d k}{(2\pi)^d} \frac{18 \, \{\partial_j D(k)\}^2}{\{1 - D(k)\}^3}. \end{aligned}$$

In Appendix B of ref. 6 and Appendix A of ref. 9, it has been proved that the right side is bounded by a d-independent multiple of $(d-4)^{-1}$ for the nearest-neighbor model, or by an L-independent multiple of $L^{2-d} (\ln L)^d$ for the spread-out model. The contribution to $\mathscr{W}_G(X)$ with $X \neq O$ from the second term of (3.24) is equal to the Fourier transform of

$$\varepsilon \sum_{Z} G(X, Z) \sum_{Y, O'} G(O, Y) \frac{\varepsilon \, \mathscr{I}_{O, O'}}{|\mathscr{I}|} \, \|\sigma(O')\|^2 \, G(Y + O', Z).$$

This is bounded by

$$\begin{aligned} &\varepsilon^2 \sum_{Y, Z, O'} \frac{\mathscr{I}_{O, O'}}{|\mathscr{I}|} \, \|\sigma(O')\|^2 \Big\{ \{G(X, Z) \, G(O', Y + O') \, G(Y + O', Z) \\ &\qquad - \Delta_{X, Z} \, \Delta_{O', Y + O'} \, \Delta_{Y + O', Z}\} + \Delta_{X, Z} \, \Delta_{O', Y + O'} \, \Delta_{Y + O', Z} \Big\} \\ &\qquad \leqslant \sum_{O'} \frac{\mathscr{I}_{O, O'}}{|\mathscr{I}|} \, \|\sigma(O')\|^2 \, \{\mathscr{T}_G(X - O') + \delta_{\sigma(X), \sigma(O')}\} \leqslant |\mathscr{V}| \, \bar{\mathscr{T}}_G + \bar{\mathscr{V}}, \end{aligned} \tag{3.25}$$

where $\bar{\mathcal{T}}_G = \sup_X \mathcal{T}_G(X)$ and

$$|\mathcal{V}| = \sum_X \frac{\mathcal{I}_{O,X}}{|\mathcal{I}|} \|\sigma(X)\|^2, \qquad \bar{\mathcal{V}} = \sup_X \frac{\mathcal{I}_{O,X}}{|\mathcal{I}|} \|\sigma(X)\|^2. \tag{3.26}$$

Since $|\mathcal{V}| = 1$ and $\bar{\mathcal{V}} = (2d)^{-1}$ for the nearest-neighbor model and $|\mathcal{V}| = \mathcal{O}(L^2)$ and $\bar{\mathcal{V}} = \mathcal{O}(L^{2-d})$ for the spread-out model, the right side of (3.25) is $\mathcal{O}(d^{-1})$ and $\mathcal{O}(L^{(4-d)/2} (\ln L)^{d/4})$ respectively. The proof of the bound on $\mathcal{W}_G(X)$ is now completed.

4. THE LACE EXPANSION

We use the lace expansion to investigate the infrared behavior of the connectivity function. The lace expansion produces a convolution equation (4.1) for the connectivity function. The expansion can be derived through the inclusion-exclusion approach or the algebraic method. Both derivations lead to the same result.

First we make several definitions. For a bond set $\mathbf{B}$, X is said to be connected to Y *off* $\mathbf{B}$ if X is connected to Y without using bonds in $\mathbf{B}$. We define $\mathbf{C}_{(U,V)}(X)$ for the set of sites connected from X off (U, V). A bond (U, V) is said to be *pivotal* for the connection from X to Y if $X \to U$ and $V \to Y \notin \mathbf{C}_{(U,V)}(X)$. We define $\mathbf{B}_{\text{piv}}(X, Y)$ for the set of pivotal bonds for the connection from X to Y; we note that $\mathbf{B}_{\text{piv}}(X, Y)$ is a random set depending on a configuration $\mathscr{C}$, which is a collection of states of bonds. X is said to be *doubly connected* to Y if $X \to Y$ and $\mathbf{B}_{\text{piv}}(X, Y) = \varnothing$; we write $X \rightrightarrows Y$ for this event. We define $E_{(V,X;\mathbf{C})}$ for $\mathbf{C} \subset \mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$ to be the event satisfying the following conditions:

- there is at least one open path from V to X passing through $\mathbf{C}$,
- either $\mathbf{B}_{\text{piv}}(V, X) = \varnothing$ or there are no open paths passing through $\mathbf{C}$ from V to the bottom of the last bond of $\mathbf{B}_{\text{piv}}(V, X) \neq \varnothing$.

Proposition 4.1 (The Lace Expansion). For $N \geqslant 0$,

$$\varphi(O, X) = \Gamma_N(O, X) + \sum_{(U,V)} \Gamma_N(O, U)\, p(U, V)\, \varphi(V, X) \\ + (-1)^{N+1} R_{N+1}(O, X), \tag{4.1}$$

where $\Gamma_N(O, X) = \sum_{n=0}^N (-1)^n g_n(O, X)$. The irreducible two-point function $g_n(O, X)$ is defined as

$$g_n(O,X) = \begin{cases} \mathbb{P}(O \rightrightarrows X), & \text{for} \quad n=0, \\ \sum_{(U_1,V_1)} p(U_1,V_1) \cdots \sum_{(U_n,V_n)} p(U_n,V_n)\, \mathbb{P}\left(O \rightrightarrows U_1, \bigcap_{i=1}^{n} E_{(V_i,U_{i+1};\,\mathbf{C}_{(U_i,V_i)}(V_{i-1}))}\right), \\ \quad \text{for} \quad n \geqslant 1, \end{cases}$$

and the remainder $R_N(O,X)$ is defined as

$$R_N(O,X) = \begin{cases} \sum_{(U,V)} p(U,V)\, \mathbb{P}(O \rightrightarrows U, V \to X \in \mathbf{C}_{(U,V)}(O)), & \text{for} \quad N=1, \\ \sum_{(U_1,V_1)} p(U_1,V_1) \cdots \sum_{(U_N,V_N)} p(U_N,V_N) \\ \quad \times \mathbb{P}\left(O \rightrightarrows U_1, \bigcap_{i=1}^{N-1} E_{(V_i,U_{i+1};\,\mathbf{C}_{(U_i,V_i)}(V_{i-1}))}, V_N \to X \in \mathbf{C}_{(U_N,V_N)}(V_{N-1})\right), \\ \quad \text{for} \quad N \geqslant 2, \end{cases}$$

provided that $V_0 = O$ and $U_{n+1} = X$.

Proof. Since the lace expansion for oriented percolation has been proved in ref. 10 by the algebraic method, we prove it below by the inclusion-exclusion approach[7].

First we prove (4.1) for $N=0$. We can divide the event $O \to X$ into two disjoint cases: either $\mathbf{B}_{\mathrm{piv}}(O,X)$ is empty or not. If $O \to X$ and $\mathbf{B}_{\mathrm{piv}}(O,X) \neq \varnothing$, then there is an open bond $(U,V) \in \mathbf{B}_{\mathrm{piv}}(O,X)$ satisfying $O \rightrightarrows U$: (U,V) is the first bond of $\mathbf{B}_{\mathrm{piv}}(O,X)$, and $O \to X$ is realized if and only if (U,V) is open. Since the state of (U,V) is independent of the event that (U,V) is the first bond of $\mathbf{B}_{\mathrm{piv}}(O,X)$,

$$\begin{aligned} \varphi(O,X) &= \mathbb{P}(O \to X, \mathbf{B}_{\mathrm{piv}}(O,X) = \varnothing) + \mathbb{P}(O \to X, \mathbf{B}_{\mathrm{piv}}(O,X) \neq \varnothing) \\ &= \mathbb{P}(O \rightrightarrows X) + \sum_{(U,V)} p(U,V)\, \mathbb{P}(O \rightrightarrows U, (U,V) \in \mathbf{B}_{\mathrm{piv}}(O,X)). \end{aligned}$$

[7] Although the proof is similar to that of Proposition 2.3 of ref. 7, thanks to the construction of the models, no nested expectations as in ref. 7 appear.

By the definition of the pivotal bond and the inclusion-exclusion relation, we have

$$\begin{aligned}\mathbb{P}(O \rightrightarrows U, (U, V) \in \mathbf{B}_{\rm piv}(O, X)) &= \mathbb{P}(O \rightrightarrows U, V \to X \notin \mathbf{C}_{(U,V)}(O)) \\ &= \mathbb{P}(O \rightrightarrows U, V \to X) \\ &\quad - \mathbb{P}(O \rightrightarrows U, V \to X \in \mathbf{C}_{(U,V)}(O)).\end{aligned}$$

Since $\tau(V) = \tau(U) + \varepsilon$, $O \rightrightarrows U$ is independent of $V \to X$, and thus the first term of the right side is equal to $g_0(O, U)\, \varphi(V, X)$. Now we have obtained the result (4.1) for $N = 0$.

Next we expand $R_1(O, X)$ and prove (4.1) for $N = 1$. We divide the event $V \to X \in \mathbf{C}_{(U,V)}(O)$ into two cases: either there is or is not an open bond $(U', V') \in \mathbf{B}_{\rm piv}(V, X)$ satisfying $U' \in \mathbf{C}_{(U,V)}(O)$; if there are no such pivotal bonds, then the event $E_{(V, X;\, \mathbf{C}_{(U,V)}(O))}$ occurs. Therefore

$$\begin{aligned}&\mathbb{P}(O \rightrightarrows U, V \to X \in \mathbf{C}_{(U,V)}(O)) \\ &\quad = \mathbb{P}(O \rightrightarrows U, E_{(V, X;\, \mathbf{C}_{(U,V)}(O))}) + \sum_{(U',V')} p(U', V')\, \mathbb{P}\begin{pmatrix} O \rightrightarrows U, E_{(V, U';\, \mathbf{C}_{(U,V)}(O))}, \\ (U', V') \in \mathbf{B}_{\rm piv}(V, X) \end{pmatrix}.\end{aligned} \tag{4.2}$$

We have exploited again the fact that the state of (U', V') is independent of the event that (U', V') is the first bond of $\mathbf{B}_{\rm piv}(V, X)$ with $O \to U'$ off (U, V). As in the case of $N = 0$, it follows from the definition of the pivotal bond, from the inclusion-exclusion relation and from the Markov property, that

$$\begin{aligned}&\mathbb{P}\begin{pmatrix} O \rightrightarrows U, E_{(V, U';\, \mathbf{C}_{(U,V)}(O))}, \\ (U', V') \in \mathbf{B}_{\rm piv}(V, X) \end{pmatrix} \\ &\qquad = \mathbb{P}(O \rightrightarrows U, E_{(V, U';\, \mathbf{C}_{(U,V)}(O))})\, \varphi(V', X) - \mathbb{P}\begin{pmatrix} O \rightrightarrows U, E_{(V, U';\, \mathbf{C}_{(U,V)}(O))}, \\ V' \to X \in \mathbf{C}_{(U',V')}(V) \end{pmatrix}.\end{aligned}$$

Finally we substitute the above identity to (4.2) to obtain

$$R_1(O, X) = g_1(O, X) + \sum_{(U',V')} g_1(O, U')\, p(U', V')\, \varphi(V', X) - R_2(O, X),$$

and thus obtain the result (4.1) for $N = 1$.

The remaining expansion for $N \geqslant 2$ can be proved inductively. ▮

5. ESTIMATES

In this section, we estimate $\hat{\Gamma}_N(K)$, $\hat{R}_{N+1}(K)$ and $\hat{\Gamma}_N(O)-\hat{\Gamma}_N(K)$, which have been used in Section 3.3 to prove the infrared bound on the connectivity function.

In order to estimate $\hat{\Gamma}_N(K)$, $\hat{R}_{N+1}(K)$ and $\hat{\Gamma}_N(O)-\hat{\Gamma}_N(K)$, we use the van den Berg–Kesten inequality (the BK inequality). We begin with introducing this inequality without its proof; the proof can be found in ref. 5. We can introduce a natural partial order among configurations, denoted by $\mathscr{C} \leqslant \mathscr{C}'$, which is defined to hold if $\mathbb{1}_{\{(X,Y) \text{ is open on } \mathscr{C}\}} \leqslant \mathbb{1}_{\{(X,Y) \text{ is open on } \mathscr{C}'\}}$ for any (X, Y). An event E is said to be *increasing* if $\mathbb{1}_{\{E \text{ occurs on } \mathscr{C}\}} \leqslant \mathbb{1}_{\{E \text{ occurs on } \mathscr{C}'\}}$ holds for $\mathscr{C} \leqslant \mathscr{C}'$. An example of an increasing event is that $O \to X$ off (U, V) for some bond (U, V). For two events E_1 and E_2, we define $E_1 \circ E_2$ for the event that E_1 and E_2 occur *disjointly*: there exists a bond set $\mathbf{B}$ such that E_1 occurs by using bonds in $\mathbf{B}$ and E_2 occurs without using bonds in $\mathbf{B}$. For example,

$$\{O \to X\} \circ \{O' \to X'\} = \bigcup_{\mathbf{B}} \{\{O \to X \text{ off } \mathbf{B}\} \cap \{O' \to X' \text{ off } \mathbf{B}^c\}\},$$

where $\mathbf{B}^c$ is the complement of a bond set $\mathbf{B}$.

Proposition 5.1 (The BK Inequality).

$$\mathbb{P}(E_1 \circ E_2) \leqslant \mathbb{P}(E_1)\, \mathbb{P}(E_2),$$

holds for any increasing events E_1 and E_2.

Next we explain a potential difficulty involved in taking $\varepsilon \downarrow 0$, and how to overcome it. We have obtained the lace expansion for the connectivity function and obtained (3.7). $\hat{\Gamma}_N(K)$ is an alternating sum of $\{\hat{g}_n(K)\}_{n=0}^N$. Suppose that we naively use the BK inequality to estimate the irreducible two-point functions as in estimating those for discrete models. Then we have, for example,

$$|\hat{g}_0(K)| \leqslant \varepsilon \sum_X \varphi(O, X)^2 = \varepsilon + \varepsilon \sum_X \psi(O, X)^2, \tag{5.1}$$

where $\psi(O, X) = \mathbb{P}(O \rightsquigarrow X)$ and $O \rightsquigarrow X$ means that there exists a non-zero open path from O to X. The latter term of the right side is $\mathcal{O}(1)$; what is worse, $|\hat{g}_n(K)| \leqslant \mathcal{O}(\varepsilon^{-2n})$ for $n \geqslant 1$ (see Section 5.1.2). Therefore we can not obtain a meaningful upper bound on $|\hat{\Gamma}_N(K)|$, and hence the infrared behavior for the connectivity function can not be seen. This is due to the fact that the number of summations is much larger than that of factors of ε. This is the difficulty involved in naive estimates in $\varepsilon \downarrow 0$.

The key observation to overcome the above difficulty is that we can derive a factor of ε from a point where at least two disjoint open paths leave or enter. This idea is an extension of the idea used in ref. 1 to extract correct factors of ε. Let us consider the case of $g_0(O, X) = \mathbb{P}(O \rightrightarrows X)$ with $X \neq O$ for example. This is the probability that there exist at least two non-zero disjoint open paths from O to X. Since at most one temporal bond grows out of every site, at least one of two non-zero disjoint open paths goes out of O with a *spatial* bond, and at least one of those comes into X with another *spatial* bond. Thanks to this close observation and the BK inequality, we obtain in the proof of Lemma 5.2

$$|\hat{g}_0(K)| \leqslant \varepsilon + \lambda^2 \varepsilon^3 \sum_{X, O', X'} \mathscr{I}_{O,O'} \mathscr{I}_{X',X} \times \{\psi(O, X)\, \varphi(O', X') + \varphi(O', X)\, \psi(O, X')\}. \tag{5.2}$$

The latter term of the right side is $\mathcal{O}(\varepsilon^2)$ in contrast with (5.1). We have thus achieved deriving extra factors of ε, which enable us to control a limit $\varepsilon \downarrow 0$.

At the end of the introduction of this section, we define several connection notations.

$$\{O \rightarrowtail X\} = \{(O, \Theta) \text{ is open}\} \circ \{\Theta \to X\},$$

$$\{O \Rightarrow\!\!\to X\} = \bigcup_{O':\sigma(O') \neq o} \{\{(O, O') \text{ is open}\} \circ \{O' \to X\}\},$$

$$\{O \twoheadrightarrow X\} = \{O \to X - \Theta\} \circ \{(X - \Theta, X) \text{ is open}\},$$

$$\{O \to\!\!\Rightarrow X\} = \bigcup_{X':\sigma(X') \neq \sigma(X)} \{\{O \to X'\} \circ \{(X', X) \text{ is open}\}\},$$

$$\{O \rightarrowtail\!\!\!\!\twoheadrightarrow X\} = \{(O, \Theta) \text{ is open}\} \circ \{\Theta \twoheadrightarrow X\}.$$

We organize the rest of this section as follows. $\hat{\Gamma}_N(K)$ is estimated in Section 5.1. Since $\hat{\Gamma}_N(K)$ is an alternating sum of $\{\hat{g}_n(K)\}_{n=0}^N$, Section 5.1 is mainly devoted to estimating $\hat{g}_n(K)$. These estimates are directly used to bound $|\hat{R}_N(K)|$ in Section 5.2. Finally in Section 5.3, we estimate $\hat{\Gamma}_N(O) - \hat{\Gamma}_N(K)$.

5.1. Estimate of $\hat{\Gamma}_N(K)$ for $N \geqslant 0$

5.1.1. Estimate of $\hat{g}_0(K)$

To estimate $\hat{g}_n(K)$ for $n \geqslant 0$, we use the following inequalities:

$$\varepsilon \sum_{Y, O', O''} \frac{\mathscr{I}_{O,O'}}{|\mathscr{I}|} \frac{\mathscr{I}_{O,O''}}{|\mathscr{I}|} \varphi(O', Y)\, \varphi(X+O'', Y) \leqslant 2\, \mathscr{K}(X), \tag{5.3}$$

$$\varepsilon \sum_{Y} \{\varphi(O, Y)\, \varphi(X, Y) - \Delta_{O,Y}\, \Delta_{X,Y}\} \leqslant 2\, \mathscr{T}(X). \tag{5.4}$$

The proof of these inequalities is the same as that of (2.20) in ref. 1.

Lemma 5.2. Under the inequalities in $\mathfrak{P}_4$,

$$|\hat{g}_0(K)| \leqslant \varepsilon \sum_{X} g_0(O, X) \leqslant \varepsilon + C\, \kappa\, \varepsilon^2, \tag{5.5}$$

holds for some ε-independent finite constant C. This proves (3.8) for $n = 0$.

Proof. We begin with proving the inequality

$$g_0(O, X) \leqslant \delta_{O,X} + \sum_{O', X'} (\lambda\, \varepsilon)^2\, \mathscr{I}_{O,O'}\, \mathscr{I}_{X',X} \times \{\psi(O, X)\, \varphi(O', X') + \varphi(O', X)\, \mathbb{P}(O \rightarrowtail X')\}. \tag{5.6}$$

If X ($\neq O$) is the arrival point of two non-zero disjoint open paths, then at least one of those paths enters X with a spatial bond. The contribution from this case is bounded by

$$\mathbb{P}(\{O \rightsquigarrow X\} \circ \{O \longrightarrow\!\!\!\!\Rightarrow X\}) \leqslant \sum_{X' : \sigma(X') \neq \sigma(X)} \mathbb{P}(\{O \rightsquigarrow X\} \circ \{O \to X'\} \circ \{(X', X) \text{ is open}\}). \tag{5.7}$$

By the construction of the models, there is no contribution from the case of $O = X'$ to the right side. If two non-zero disjoint open paths emanate from O, then at least one of them leaves O with a spatial bond. Therefore we use the BK inequality to bound the right side of (5.7) by

$$\sum_{X'} \lambda\, \varepsilon\, \mathscr{J}_{X',X}\, \mathbb{P}(\{O \rightsquigarrow X\} \circ \{O \rightsquigarrow X'\})$$

$$\leqslant \sum_{X'} \lambda\, \varepsilon\, \mathscr{J}_{X',X}\, \{\mathbb{P}(\{O \rightsquigarrow X\} \circ \{O \Rightarrow\!\!\rightarrow X'\}) + \mathbb{P}(\{O \Rightarrow\!\!\rightarrow X\} \circ \{O \rightarrowtail X'\})\}$$

$$\leqslant \sum_{O',X'} (\lambda\, \varepsilon)^2\, \mathscr{J}_{O,O'}\, \mathscr{J}_{X',X}\, \{\psi(O,X)\, \varphi(O',X') + \varphi(O',X)\, \mathbb{P}(O \rightarrowtail X')\},$$

and thus obtain the inequality (5.6).

Using (5.3) and (5.6), we can bound $|\hat{g}_0(K)|$ by

$$\varepsilon + (\lambda\, |\mathscr{J}|)^2\, \varepsilon^3 \sum_{O',O'',X} \frac{\mathscr{J}_{O,O'}}{|\mathscr{J}|}\, \frac{\mathscr{J}_{O,O''}}{|\mathscr{J}|}$$

$$\times \{\psi(O,X)\, \varphi(O', X-O'') + \varphi(O',X)\, \mathbb{P}(O \rightarrowtail X-O'')\}$$

$$\leqslant \varepsilon + 4\, (\lambda\, |\mathscr{J}|)^2\, \bar{\mathscr{K}}\, \varepsilon^2, \tag{5.8}$$

and thus obtain (5.5) under the inequalities in $\mathfrak{P}_4$. ▮

5.1.2. Estimate of $\hat{g}_n(K)$ for $n \geqslant 1$

In this section, we estimate $\hat{g}_n(K)$ for $n \geqslant 1$. We present in Lemma 5.3 an increasing event, of which the event in the definition of $g_n(O,X)$ is a subset. Before using the BK inequality to this increasing event, we must pay attention to points in diagrams where two disjoint open paths leave or enter, as in the estimate of $\hat{g}_0(K)$. Thanks to this close observation and the BK inequality, we can obtain a bound on $|\hat{g}_n(K)|$ in Lemma 5.4, which enables us to take a meaningful limit in $\varepsilon \downarrow 0$.

First we present the increasing event stated above.

Lemma 5.3. For $n \geqslant 1$,

$$\left\{O \rightrightarrows U_1,\ \bigcap_{i=1}^{n} E_{(V_i,\, U_{i+1};\, \mathbf{C}_{(U_i,V_i)}(V_{i-1}))}\right\}$$

$$\subset \bigcup_{W_1,\ldots,W_n} \{L_{(O,W_1,U_1)} \circ \{M^{(U_1,V_1)}_{(W_1,V_1;W_2,U_2)} \cup N^{(U_1,V_1)}_{(W_1,V_1;W_2,U_2)}\} \circ \cdots$$

$$\cdots \circ \{M^{(U_{n-1},V_{n-1})}_{(W_{n-1},V_{n-1};W_n,U_n)} \cup N^{(U_{n-1},V_{n-1})}_{(W_{n-1},V_{n-1};W_n,U_n)}\} \circ R^{(U_n,V_n)}_{(W_n,V_n,X)}\}, \tag{5.9}$$

provided that $V_0 = O$ and $U_{n+1} = X$, where

$$L_{(O,W,U)} = \{O \to W\} \circ \{O \to U\} \circ \{W \to U\},$$

$$R^{(U,V)}_{(W,V,X)} = \{W \rightsquigarrow X \text{ off } (U,V)\} \circ \{V \to X\},$$

$$M^{(U,V)}_{(W,V;W',U')} = \{W \rightsquigarrow U' \text{ off } (U,V)\} \circ \{V \to W'\} \circ \{W' \to U'\},$$

$$N^{(U,V)}_{(W,V;W',U')} = \{W \rightsquigarrow W' \text{ off } (U,V)\} \circ \{V \to U'\} \circ \{W' \to U'\}.$$

The event $R^{(U,V)}_{(W,V,X)}$ is defined only when $\tau(W) \leqslant \tau(U) < \tau(V) \leqslant \tau(X)$, and the events $M^{(U,V)}_{(W,V;W',U')}$ and $N^{(U,V)}_{(W,V;W',U')}$ are defined only when $\tau(W) \leqslant \tau(U) < \tau(V) \leqslant \tau(W') \leqslant \tau(U')$.

We omit the proof of the lemma because we can prove it as explained in ref. 10.

Next we use the above lemma to estimate $\hat{g}_n(K)$. If we naively use the BK inequality, then

$$\begin{aligned}
|\hat{g}_n(K)| \leqslant & \sum_{W_1, U_1} \mathbb{P}(L_{(O,W_1,U_1)}) \sum_{V_1, W_2, U_2} p(U_1, V_1) \\
& \times \{\mathbb{P}(M^{(U_1,V_1)}_{(W_1,V_1;W_2,U_2)}) + \mathbb{P}(N^{(U_1,V_1)}_{(W_1,V_1;W_2,U_2)})\} \cdots \\
& \cdots \sum_{V_{n-1}, W_n, U_n} p(U_{n-1}, V_{n-1}) \{\mathbb{P}(M^{(U_{n-1},V_{n-1})}_{(W_{n-1},V_{n-1};W_n,U_n)}) \\
& + \mathbb{P}(N^{(U_{n-1},V_{n-1})}_{(W_{n-1},V_{n-1};W_n,U_n)})\} \times \varepsilon \sum_{V_n, X} p(U_n, V_n)\, \mathbb{P}(R^{(U_n,V_n)}_{(W_n,V_n,X)}) \\
\leqslant & \{|p|\, \varepsilon^{-2}\}^n \left\{ \varepsilon^2 \sum_{W,U} \varphi(O,W)\, \varphi(O,U)\, \varphi(W,U) \right\} \\
& \times \left\{ \sup_{W,V} \varepsilon \sum_X \psi(W,X)\, \varphi(V,X) \right\} \\
& \times \left\{ \sup_{W,V} \varepsilon^2 \sum_{W',U'} \{\psi(W,U')\, \varphi(V,W') + \psi(W,W')\, \varphi(V,U')\} \right. \\
& \left. \times \varphi(W'\, U') \right\}^{n-1},
\end{aligned}$$

and hence $|\hat{g}_n(K)| \leqslant \mathcal{O}(\varepsilon^{-2n})$. In the diagram considered, however, there are $2n+2$ points where two disjoint open paths leave or enter. Thanks to this close observation, we thus obtain $|\hat{g}_n(K)| \leqslant \mathcal{O}(\varepsilon^2)$, as explained in the following lemma.

Lemma 5.4. For $n \geqslant 1$, under the inequalities in $\mathfrak{P}_4$,

$$|\hat{g}_n(K)| \leqslant \varepsilon \sum_X g_n(O, X) \leqslant C\, \kappa\, (C'\, \mu)^{n-1}\, \varepsilon^2, \tag{5.10}$$

holds for some ε-independent finite constants C and C'. This proves (3.8) for $n \geqslant 1$.

Proof. First we consider the case of $n = 1$. We use Lemma 5.3, the Markov property and the BK inequality to bound $g_1(O, X)$ by

$$\begin{aligned}
&\sum_{W,(U,V)} p(U,V)\, \mathbb{P}(L_{(O,W,U)} \circ R^{(U,V)}_{(W,V,X)})\, \{\delta_{W,U} + (1-\delta_{W,U})\} \\
&\leqslant \sum_{W,(U,V)} p(U,V) \Big\{ g_0(O,U)\, \delta_{W,U}\, \mathbb{P}(R^{(U,V)}_{(U,V,X)}) + \mathbb{P}({}^{\sigma}L_{(O,W,U)} \circ R^{(U,V)}_{(W,V,X)}) \\
&\quad + \sum_{W'} \mathbb{P}({}^{\tau}L_{(O,W,U)} \circ \{(W,W') \text{ is open}\} \circ R^{(U,V)}_{(W',V,X)}) \Big\} \\
&\leqslant \sum_{W,U} \Big\{ g_0(O,U)\, \delta_{W,U} \sum_V p(U,V)\, \mathbb{P}(R^{(U,V)}_{(U,V,X)}) \\
&\quad + \mathbb{P}({}^{\sigma}L_{(O,W,U)}) \sum_V p(U,V)\, \mathbb{P}(R^{(U,V)}_{(W,V,X)}) \\
&\quad + \mathbb{P}({}^{\tau}L_{(O,W,U)}) \sum_{W',V} \lambda\, \varepsilon\, \mathscr{I}_{W,W'}\, p(U,V)\, \mathbb{P}(R^{(U,V)}_{(W',V,X)}) \Big\},
\end{aligned} \tag{5.11}$$

where

$$\begin{aligned}
{}^{\sigma}L_{(O,W,U)} &= \{O \to W\} \circ \{O \rightsquigarrow U\} \circ \{W \Rightarrow\!\!\to U\}, \\
{}^{\tau}L_{(O,W,U)} &= \{O \to W\} \circ \{O \rightsquigarrow U\} \circ \{W \rightarrowtail U\}.
\end{aligned}$$

In order to estimate $\sum_{V,X} p(U,V)\, \mathbb{P}(R^{(U,V)}_{(W,V,X)})$, we must pay attention to the point X where two disjoint open paths enter in the diagram; we must observe the point W as well, if $W = U$. Using repeatedly the BK inequality and the inequalities

$$\begin{aligned}
&(1-\varepsilon)\,\varphi(U+\Theta, X') = \mathbb{P}(U \rightarrowtail X'),\\
&(1-\varepsilon)\,\mathbb{P}(U+\Theta \twoheadrightarrow X) = \mathbb{P}(U \rightarrowtail\!\!\!\!\twoheadrightarrow X),\\
&\delta_{U,X'} + \mathbb{P}(U \rightarrowtail X') \leqslant \varphi(U, X'),\\
&(1-\varepsilon)\,\delta_{U,X-\Theta} + \mathbb{P}(U \rightarrowtail\!\!\!\!\twoheadrightarrow X) \leqslant \mathbb{P}(U \twoheadrightarrow X),
\end{aligned} \tag{5.12}$$

we obtain

$$\sum_{V,X} p(U,V)\,\mathbb{P}(R^{(U,V)}_{(W,V,X)}) \leqslant \varepsilon\,\bar{\mathscr{R}}_1\,\mathbb{1}_{\{W=U\}} + \bar{\mathscr{R}}_2\,\mathbb{1}_{\{W\neq U\}},$$

where, under the inequalities in $\mathfrak{P}_4$,

$$\begin{aligned}
&\bar{\mathscr{R}}_1 \equiv 8\,(\lambda\,|\mathscr{J}|)^2\,\bar{\mathscr{K}} \leqslant C_1\,\kappa,\\
&\bar{\mathscr{R}}_2 \equiv 2\,\lambda\,|\mathscr{J}|\,(2\,\bar{\mathscr{T}} + \bar{\mathscr{J}}/|\mathscr{J}| + 2\,\lambda\,|\mathscr{J}|\,\bar{\mathscr{K}}\,\varepsilon) \leqslant C_2\,\mu,
\end{aligned} \tag{5.13}$$

for some constants C_1 and C_2, because $\bar{\mathscr{J}} \equiv \sup_X \mathscr{J}_{O,X}$ is equal to 1 for the nearest-neighbor model or to $\{(2\,L+1)^d - 1\}^{-1}$ for the spread-out model such that $\bar{\mathscr{J}}/|\mathscr{J}| \leqslant \kappa$ holds for both models.

We only prove the latter inequality of (5.13); the former can be proved similarly.

$$\begin{aligned}
&\sum_V p(U,V)\,\mathbb{P}(R^{(U,V)}_{(W,V,X)})\,\{\delta_{V,X} + (1-\delta_{V,X})\}\\
&\quad\leqslant \{\lambda\,\varepsilon\,\mathscr{J}_{U,X}\,\psi(W,X) + (1-\varepsilon)\,\delta_{U,X-\Theta}\,\mathbb{P}(W \Longrightarrow X)\}\,\delta_{V,X}\\
&\qquad + \sum_V \{(1-\varepsilon)\,\delta_{U,V-\Theta} + \lambda\,\varepsilon\,\mathscr{J}_{U,V}\}\\
&\qquad \times \{\mathbb{P}(\{W \rightsquigarrow X\} \circ \{V \Longrightarrow X\}) + \mathbb{P}(\{W \Longrightarrow X\} \circ \{V \twoheadrightarrow X\})\}\\
&\qquad\qquad (\because \text{the close observation,})\\
&\quad\leqslant \lambda\,\varepsilon\,\mathscr{J}_{U,X}\,\psi(W,X) + (1-\varepsilon)\,\delta_{U,X-\Theta} \sum_{X'} \lambda\,\varepsilon\,\mathscr{J}_{X',X}\,\varphi(W,X')\\
&\qquad + \sum_{X'} (1-\varepsilon)\,\lambda\,\varepsilon\,\mathscr{J}_{X',X}\,\{\psi(W,X)\,\varphi(U+\Theta, X')\\
&\qquad + \varphi(W,X')\,\mathbb{P}(U+\Theta \twoheadrightarrow X)\}\\
&\qquad + \sum_{V,X'} (\lambda\,\varepsilon)^2\,\mathscr{J}_{U,V}\,\mathscr{J}_{X',X}\,\{\psi(W,X)\,\varphi(V,X') + \varphi(W,X')\,\mathbb{P}(V \twoheadrightarrow X)\}.\\
&\qquad\qquad (\because \text{the BK inequality.})
\end{aligned} \tag{5.14}$$

By using the inequalities in (5.12), the first three terms of the right side of (5.14) can be put together, and (5.14) is bounded by

$$\sum_{X'} \lambda\, \varepsilon\, \mathscr{J}_{X',X} \Big\{ \psi(W,X)\, \varphi(U,X') + \varphi(W,X')\, \mathbb{P}(U \twoheadrightarrow X) \\ + \sum_{V} \lambda\, \varepsilon\, \mathscr{J}_{U,V}\, \{\psi(W,X)\, \varphi(V,X') + \varphi(W,X')\, \mathbb{P}(V \twoheadrightarrow X)\} \Big\}.$$

We obtain the latter of (5.13) by summing the above expression over X and using (5.3) and (5.4).

After defining $\overline{\mathscr{R}}_3 = \lambda\, |\mathscr{J}|\, \overline{\mathscr{R}}_2$ and $\overline{\mathscr{L}}_1 = 1 + 4\, (\lambda\, |\mathscr{J}|)^2\, \overline{\mathscr{K}}\, \varepsilon$, which follows from (5.8), we obtain

$$|\hat{g}_1(K)| \leqslant \overline{\mathscr{L}}_1\, \overline{\mathscr{R}}_1\, \varepsilon^2 + \varepsilon \sum_{W,U} \mathbb{P}({}^{\sigma}L_{(O,W,U)})\, \overline{\mathscr{R}}_2 + \varepsilon^2 \sum_{W,U} \mathbb{P}({}^{\tau}L_{(O,W,U)})\, \overline{\mathscr{R}}_3.$$

We can also estimate $\sum_{W,U} \mathbb{P}({}^{\sigma}L_{(O,W,U)})$ and $\sum_{W,U} \mathbb{P}({}^{\tau}L_{(O,W,U)})$ in the same way as in proving the inequalities in (5.13); we must pay attention to the points O and U to extract correct factors of ε.

$$\sum_{W,U} \mathbb{P}({}^{\sigma}L_{(O,W,U)}) \leqslant 4\, (\lambda\, |\mathscr{J}|)^2\, (2 + \lambda\, |\mathscr{J}|)\, \overline{\mathscr{K}}\, \varepsilon \equiv \overline{\mathscr{L}}_2\, \varepsilon,$$

$$\sum_{W,U} \mathbb{P}({}^{\tau}L_{(O,W,U)}) \leqslant 4\, (\lambda\, |\mathscr{J}|)^2\, (1 + \varepsilon)\, \overline{\mathscr{K}} \equiv \overline{\mathscr{L}}_3.$$

Under the inequalities in $\mathfrak{P}_4$, $\overline{\mathscr{L}}_2$ and $\overline{\mathscr{L}}_3$ are bounded by $C_3\, \kappa$ for some constant C_3. This completes the proof of the inequality (5.10) for $n = 1$.

Next we consider the case of $n = 2$; the case of $n \geqslant 3$ can be proved by induction. Following the way of deriving (5.11), we can bound $|\hat{g}_2(K)|$ by

$$\begin{aligned} \varepsilon \sum_{\substack{X, W, (U,V), \\ W', (U',V')}} & p(U,V)\, p(U',V')\, \{\mathbb{P}(L_{(O,W,U)} \circ M^{(U,V)}_{(W,V;W',U')} \circ R^{(U',V')}_{(W',V',X)}) \\ & + \mathbb{P}(L_{(O,W,U)} \circ N^{(U,V)}_{(W,V;W',U')} \circ R^{(U',V')}_{(W',V',X)})\}\{\delta_{W',U'} + (1 - \delta_{W',U'})\} \\ \leqslant \varepsilon^2 \sum_{W',U'} & \{2\, g_1(O,U')\, \delta_{W',U'}\, \overline{\mathscr{R}}_1 + \varepsilon^{-1}\, \{\mathscr{P}^{\sigma M}_{(O,W',U')} + \mathscr{P}^{\sigma N}_{(O,W',U')}\}\, \overline{\mathscr{R}}_2 \\ & + \{\mathscr{P}^{\tau M}_{(O,W',U')} + \mathscr{P}^{\tau N}_{(O,W',U')}\}\, \overline{\mathscr{R}}_3\}, \end{aligned} \tag{5.15}$$

where

$$\mathscr{P}^{\sigma M}_{(O,W',U')} = \sum_{W,(U,V)} p(U,V)\,\mathbb{P}(L_{(O,W,U)} \circ {}^{\sigma}M^{(U,V)}_{(W,V;W',U')}),$$

$$\mathscr{P}^{\sigma N}_{(O,W',U')} = \sum_{W,(U,V)} p(U,V)\,\mathbb{P}(L_{(O,W,U)} \circ {}^{\sigma}N^{(U,V)}_{(W,V;W',U')}),$$

$$\mathscr{P}^{\tau M}_{(O,W',U')} = \sum_{W,(U,V)} p(U,V)\,\mathbb{P}(L_{(O,W,U)} \circ {}^{\tau}M^{(U,V)}_{(W,V;W',U')}),$$

$$\mathscr{P}^{\tau N}_{(O,W',U')} = \sum_{W,(U,V)} p(U,V)\,\mathbb{P}(L_{(O,W,U)} \circ {}^{\tau}N^{(U,V)}_{(W,V;W',U')}),$$

and where

$${}^{\sigma}M^{(U,V)}_{(W,V;W',U')} = \{W \rightsquigarrow U' \text{ off } (U,V)\} \circ \{V \to W'\} \circ \{W' \Rightarrow\!\!\!\to U'\},$$

$${}^{\sigma}N^{(U,V)}_{(W,V;W',U')} = \{W \rightsquigarrow W' \text{ off } (U,V)\} \circ \{V \rightsquigarrow U'\} \circ \{W' \Rightarrow\!\!\!\to U'\},$$

$${}^{\tau}M^{(U,V)}_{(W,V;W',U')} = \{W \rightsquigarrow U' \text{ off } (U,V)\} \circ \{V \to W'\} \circ \{W' \rightarrowtail U'\},$$

$${}^{\tau}N^{(U,V)}_{(W,V;W',U')} = \{W \rightsquigarrow W' \text{ off } (U,V)\} \circ \{V \rightsquigarrow U'\} \circ \{W' \rightarrowtail U'\}.$$

These events are defined only when $\tau(W) \leqslant \tau(U) < \tau(V) \leqslant \tau(W') \leqslant \tau(U')$ because of the definitions of $M^{(U,V)}_{(W,V;W',U')}$ and $N^{(U,V)}_{(W,V;W',U')}$.

We again follow the way of deriving (5.11) to bound $\mathscr{P}^{\sigma M}_{(O,W',U')}$ by

$$\sum_{W,U} \Big\{ g_0(O,U)\,\delta_{W,U} \sum_{V} p(U,V)\,\mathbb{P}({}^{\sigma}M^{(U,V)}_{(U,V;W',U')})$$
$$+\,\mathbb{P}({}^{\sigma}L_{(O,W,U)}) \sum_{V} p(U,V)\,\mathbb{P}({}^{\sigma}M^{(U,V)}_{(W,V;W',U')})$$
$$+\,\mathbb{P}({}^{\tau}L_{(O,W,U)}) \sum_{W'',V} \lambda\,\varepsilon\,\mathscr{I}_{W,W''}\,p(U,V)\,\mathbb{P}({}^{\sigma}M^{(U,V)}_{(W'',V;W',U')})\Big\}. \tag{5.16}$$

Following the way of proving (5.13) by paying attention to the point U' where two disjoint open paths enter in the diagram (it is necessary to observe the point W as well, if $W = U$), we obtain

$$\sum_{V,W',U'} p(U,V)\,\mathbb{P}({}^{\sigma}M^{(U,V)}_{(W,V;W',U')}) \leqslant \varepsilon\,\bar{\mathscr{H}}_{12}\,\mathbb{1}_{\{W=U\}} + \bar{\mathscr{H}}_{22}\,\mathbb{1}_{\{W\neq U\}},$$

where, under the inequalities in $\mathfrak{P}_4$,

$$\bar{\mathscr{H}}_{12} \equiv 4\,(\lambda\,|\mathscr{J}|)^2\,(1+\lambda\,|\mathscr{J}|)\;\bar{\mathscr{K}} \leqslant C_4\,\kappa,$$
$$\bar{\mathscr{H}}_{22} \equiv \lambda\,|\mathscr{J}|\,|p|\,(2\,\bar{\mathscr{T}} + \bar{\mathscr{J}}/|\mathscr{J}| + 2\,\lambda\,|\mathscr{J}|\;\bar{\mathscr{K}}) \leqslant C_5\,\mu,$$

for some constants C_4 and C_5. After defining $\bar{\mathscr{H}}_{32} = \lambda\,|\mathscr{J}|\;\bar{\mathscr{H}}_{22}$, we thus have

$$\varepsilon \sum_{W',U'} \mathscr{P}^{\sigma M}_{(O,W',U')}\,\bar{\mathscr{R}}_2 \leqslant \varepsilon^2 \sum_{i=1}^{3} \bar{\mathscr{L}}_i\,\bar{\mathscr{H}}_{i2}\,\bar{\mathscr{R}}_2 \leqslant C\,\kappa\,\mu\,\varepsilon^2,$$

for some ε-independent finite constant C.

We can also estimate the quantities concerning $\mathscr{P}^{\sigma N}_{(O,W',U')}$, $\mathscr{P}^{\tau M}_{(O,W',U')}$ and $\mathscr{P}^{\tau N}_{(O,W',U')}$ as

$$\varepsilon \sum_{W',U'} \mathscr{P}^{\sigma N}_{(O,W',U')}\,\bar{\mathscr{R}}_2 \leqslant \varepsilon^2 \sum_{i=1}^{3} \bar{\mathscr{L}}_i\,\bar{\mathscr{H}}_{i2}\,\bar{\mathscr{R}}_2,$$

$$\left.\begin{array}{l} \varepsilon^2 \sum_{W',U'} \mathscr{P}^{\tau M}_{(O,W',U')}\,\bar{\mathscr{R}}_3 \\ \varepsilon^2 \sum_{W',U'} \mathscr{P}^{\tau N}_{(O,W',U')}\,\bar{\mathscr{R}}_3 \end{array}\right\} \leqslant \varepsilon^2 \sum_{i=1}^{3} \hat{\mathscr{L}}_i\,\bar{\mathscr{H}}_{i3}\,\hat{\mathscr{R}}_3,$$

where, under the inequalities in $\mathfrak{P}_4$,

$$\bar{\mathscr{H}}_{13} \equiv 4\,(\lambda\,|\mathscr{J}|)^2\;\bar{\mathscr{K}} \leqslant C_6\,\kappa, \qquad \bar{\mathscr{H}}_{23} \equiv 2\,\lambda\,|\mathscr{J}|\,|p|\,(\bar{\mathscr{T}} + \bar{\mathscr{J}}/|\mathscr{J}|) \leqslant C_7\,\mu,$$

and $\bar{\mathscr{H}}_{33} = \lambda\,|\mathscr{J}|\;\bar{\mathscr{H}}_{23} \leqslant 4\,C_7\,\mu$, for some constants C_6 and C_7. This completes the proof of the inequality (5.10) for $n = 2$. ∎

5.2. Estimate of $\hat{R}_N(K)$ for $N \geqslant 1$

We estimate $\hat{R}_N(K)$ in this section. By the Markov property, we obtain

$$|\hat{R}_N(K)| \leqslant \sum_{(U,V)} g_{N-1}(O,U)\,p(U,V)\,\varepsilon \sum_X \varphi(V,X) \leqslant \hat{g}_{N-1}(O)\,|p|\,\chi(\lambda)/\varepsilon,$$

which is equivalent to (3.9) used in Section 3.3. We have already seen that $\hat{g}_{N-1}(O)$ is bounded as in (5.5) and (5.10). Therefore if $\lambda < \lambda_c$, then $\chi(\lambda) < \infty$ and thus $\hat{R}_N(K)$ converges to 0 as $N \uparrow \infty$ for sufficiently large d or L.

5.3. Estimate of $\hat{\Gamma}_N(O) - \hat{\Gamma}_N(K)$ for $N \geqslant 0$

We devote this section to estimating $\mathscr{G}_n^\tau = \varepsilon \sum_X \tau(X)\, g_n(O, X)$ and $\mathscr{G}_n^\sigma = \varepsilon \sum_X \|\sigma(X)\|^2\, g_n(O, X)$ because, by the spatial symmetry of $g_n(O, X)$, $|\hat{\Gamma}_N(O) - \hat{\Gamma}_N(K)|$ is bounded by

$$\sum_{n=0}^{N} \left\{ \varepsilon \sum_X g_n(O, X)\, |1 - e^{i\omega\tau(X)}| + \varepsilon \sum_X g_n(O, X)\, |1 - \cos k \cdot \sigma(X)| \right\}$$
$$\leqslant \sum_{n=0}^{N} \left(|\omega|\, \mathscr{G}_n^\tau + \frac{\|k\|^2}{2\,d}\, \mathscr{G}_n^\sigma \right).$$

This proves the inequality (3.13).

In order to estimate $\mathscr{G}_n^\tau$, we use the inequality

$$\tau(X)\, \varphi(O, X) = \varepsilon \sum_{n=1}^{\tau(X)/\varepsilon} \psi(O, X)$$
$$\leqslant \varepsilon \sum_{n=1}^{\tau(X)/\varepsilon} \sum_{z \in \mathbb{Z}^d} \psi(O, (z, n\,\varepsilon))\, \varphi((z, n\,\varepsilon), X) = \varepsilon \sum_Z \psi(O, Z)\, \varphi(Z, X), \quad (5.17)$$

which follows from the Markov property. We note that this inequality implies equivalence between the estimate of $\mathscr{G}_n^\tau$ and that of a quantity concerning the irreducible two-point function $g_n(O, X)$ with an extra vertex.

We only estimate $\mathscr{G}_0^\sigma$ because we have to estimate it more minutely than[8] $\mathscr{G}_n^\tau$ and $\mathscr{G}_{n+1}^\sigma$ for $n \geqslant 0$ in order that the bootstrapping argument works. Using the inequalities (5.4) and (5.6), we obtain

$$\mathscr{G}_0^\sigma \leqslant \lambda^2 \varepsilon^3 \sum_{O', X', X} \mathscr{I}_{O,O'}\, \mathscr{I}_{X',X}\, \{ \|\sigma(X)\|^2\, \psi(O, X)\, \varphi(O', X')$$
$$+ 2\, \{ \|\sigma(X - O')\|^2 + \|\sigma(O')\|^2 \}\, \varphi(O', X)\, \mathbb{P}(O \rightarrowtail X') \}$$
$$\leqslant (\lambda\,\varepsilon)^2 \sum_{O', O''} \mathscr{I}_{O,O'}\, \mathscr{I}_{O,O''}\, \{ \mathscr{W}(O' + O'') + 2\, \mathscr{W}(O' - O'')$$
$$+ 2\, \|\sigma(O')\|^2\, \{ 2\, \mathscr{T}(O' - O'') + \delta_{\sigma(O'), \sigma(O'')} \} \}.$$

We estimate $\mathscr{W}(O' \pm O'')$ more minutely; we want to estimate it without using $\sup_X \mathscr{W}(X)$.

[8] We can easily obtain the bound on $\mathscr{G}_0^\tau$ in (3.14) by using the inequalities (5.6) and (5.17). We can also obtain the other bounds in (3.14) by following the way of estimating $\sum_x |x|^2\, h_n(o, x)$ in Section 3.2 of ref. 7.

$$\begin{aligned}\mathscr{W}(O'\pm O'') \leqslant\ & \varepsilon \sum_X \|\sigma(X)\|\, \|\sigma(X-O'\mp O'')\|\, \psi(O,X)\, \varphi(O'\pm O'',X)\\ &+\|\sigma(O'\pm O'')\|\, \varepsilon \sum_X \|\sigma(X)\|\, \psi(O,X)\\ &\times\{\varphi(O'\pm O'',X)-\Delta_{O'\pm O'',X}\}\\ &+\|\sigma(O'\pm O'')\|\, \varepsilon \sum_X \|\sigma(X)\|\, \psi(O,X)\, \Delta_{O'\pm O'',X}. \end{aligned} \tag{5.18}$$

By using the Schwarz inequality, the sum of the first and second terms is bounded by

$$\mathscr{W}(O)+\|\sigma(O'\pm O'')\|\, \{2\, \mathscr{T}(O)\, \mathscr{W}(O)\}^{1/2}.$$

Because of the presence of $\Delta_{O'\pm O'',X}$, the third term of (5.18) is bounded by

$$\begin{aligned}&\|\sigma(O'\pm O'')\|\, \varepsilon \sum_X \|\sigma(X)\|\, \{\varphi(O,X)-\Delta_{O,X}\}\, \Delta_{O'\pm O'',X}\\ &\quad\leqslant 2\, \|\sigma(O'\pm O'')\|^2\, \mathscr{T}(O'\pm O'').\end{aligned}$$

By using $\|\sigma(O'\pm O'')\|^n \leqslant 2^{n-1}\{\|\sigma(O')\|^n+\|\sigma(O'')\|^n\}$ for $n\geqslant 1$ and the Schwarz inequality,

$$\sum_{O',O''} \frac{\mathscr{J}_{O,O'}}{|\mathscr{J}|}\frac{\mathscr{J}_{O,O''}}{|\mathscr{J}|}\|\sigma(O'\pm O'')\|^n \leqslant 2^n \sum_{O'} \frac{\mathscr{J}_{O,O'}}{|\mathscr{J}|}\|\sigma(O')\|^n \leqslant (2\,|\mathscr{V}|^{1/2})^n,$$
$$\text{for}\quad n=1,2.$$

Therefore

$$\begin{aligned}\mathscr{G}_0^\sigma \leqslant\ & (\lambda\,|\mathscr{J}|)^2\, \{2\,|\mathscr{V}|\,(2\,\bar{\mathscr{T}}+\bar{\mathscr{J}}/|\mathscr{J}|)\\ &+3\,\{\mathscr{W}(O)+2\,\{2\,|\mathscr{V}|\,\bar{\mathscr{T}}\,\mathscr{W}(O)\}^{1/2}+8\,|\mathscr{V}|\,\bar{\mathscr{T}}\}\}\,\varepsilon^2.\end{aligned}$$

Since $|\mathscr{V}|$ defined in (3.26) equals 1 and $\zeta=d^{-1}$ for the nearest-neighbor model and $|\mathscr{V}|=\mathscr{O}(L^2)$ and $\zeta=L^2\mu$ for the spread-out model, we obtain $\mathscr{G}_0^\sigma \leqslant C\,\zeta\,\varepsilon^2$ under the inequalities in $\mathfrak{P}_4$ for some constant C depending only on $C_{\mathscr{T}}$ and $C_{\mathscr{W}}$. This proves the bound on $\mathscr{G}_0^\sigma$ in (3.14).

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